

# An Optimization of AND-OR-EXOR Three-Level Networks

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**Abstract** — In this paper, we present a design method for AND-OR-EXOR three-level networks, where a single two-input EXOR gate is used. The network realizes an exclusive-OR of two sum-of-products expressions (EX-SOP), where the two sum-of-products expressions (SOP) cannot share products. The problem is to minimize the total number of product in the two SOPs. We introduced the  $\mu$ -equivalence of logic functions to develop minimization algorithms for EX-SOPs with up to five variables. We minimized all the representative functions of NP-equivalence classes for up to five variables and found that five-variable functions require up to 9 products in minimum EX-SOPs. For  $n$ -variable functions, minimum EX-SOPs require at most  $9 \cdot 2^{n-5}$  ( $n \geq 6$ ) products. This upper bound is smaller than  $2^{n-1}$ , the upper bound for the conventional sum-of-products expressions.

**Index Terms** — Three-level network, AND-EXOR, logic minimization, spectral method, NP-equivalence,  $\mu$ -equivalence, coordinate representation, complexity.

## I. INTRODUCTION

Logic networks are usually designed by using AND and OR gates. However, it has been observed that the addition of exclusive-OR (EXOR) gates in the design often produce better networks [5, 11, 12, 15]. For example, on the average, five-variable functions require 7.46 products in minimum SOPs (sum-of-products expressions), while 6.16 products in minimum ESOPs (exclusive-OR sum-of-products expressions) [13]. To realize an arbitrary function of six variables, minimum SOPs require up to 32 products, while minimum ESOPs require up to 15 products [8]. These reveal the advantages of designing logic networks using EXOR gates. In these designs, EXOR gates with unlimited fan-in are used. However, in most technologies, EXOR gates with many inputs are expensive.

In this paper, we present a design method for AND-OR-EXOR three-level networks. The network realizes an exclusive-OR of two sum-of-products expressions (EX-

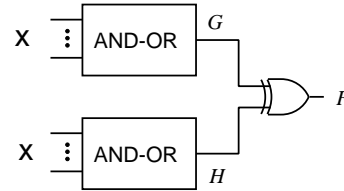


Fig. 1. AND-OR-EXOR three-level network.

SOP), where only a single two-input EXOR gate is used. Such a network is shown in Fig. 1. An EX-SOP for a function  $f$  can be written as  $F = G \oplus H$ , where  $G$  and  $H$  are SOPs. Our objective is to reduce the total number of products in  $G$  and  $H$ , where  $G$  and  $H$  cannot share products.

AND-OR-EXOR three-level network is suitable for implementing arithmetic functions. For example, the Texas Instruments SN181 arithmetic circuit has EXOR gates in the outputs [18]. Programmable logic arrays (PLAs) with two-input EXOR gates in the outputs efficiently realize adders [17]. AND-OR-EXOR is one of the simplest architecture, since it contains only a single two-input EXOR gate. However, its logic capability is quite high. Because of these, various PLAs with two-input EXOR gates in the outputs were developed. Especially, RICOH, Lattice and AMD (MMI) produced series of such PLAs.

Design methods for AND-OR-EXOR three-level networks were considered in the past [4, 16], but no practical algorithm was reported. A cut-and-try method was reported in [10] and several heuristic algorithms to simplify EX-SOPs were presented in [14]. Upper bounds on the number of products in AND-OR-EXOR expansion was reported in [3].

The paper is organized as follows: Section II introduces first the terminologies used and develops the concept of  $\mu$ -equivalence of logic functions. Section III presents the key idea for the minimization, a technique to reduce the search space, and an algorithm to minimize EX-SOPs. Section IV shows how the  $\mu$ -equivalence of logic functions

can be used to reduce the computation time for EX-SOPs and presents a minimization algorithm for EX-SOPs with five variables. Section V outlines an idea to simplify EX-SOPs with six or more variables. Section VI shows the experimental results. The last section presents conclusions and comments.

## II. DEFINITIONS AND BASIC PROPERTIES

In this section, we introduce first several notations and define the NP-equivalence classes of functions. We consider then the modified coordinate representation of functions. By using this representation, we develop the concept of  $\mu$ -equivalence of logic functions and illustrate its properties, which would be useful in the minimization of EX-SOPs with five variables. In this paper, we distinguish functions and their expressions. We use lower case letters, such as  $f, g, h$ , to represent functions, and upper case letters, such as  $F, G, H$ , to represent expressions of function.

**Definition 2.1:**  $\tau(F)$  denotes the number of products in an expression  $F$ . An expression  $F$  is said to be minimum when  $\tau(F)$  is minimum.

**Definition 2.2:**  $\tau(EX-SOP : f)$  denotes the total number of products in a minimum EX-SOP for  $f$ .  $\tau(SOP : f)$  denotes the number of products in a minimum SOP for  $f$ .

Let a function  $f$  be represented as follows:

$$f = g \oplus h. \quad (2.1)$$

Note that  $g$  and  $h$  correspond to  $G$  and  $H$  in Fig. 1, respectively. To compute  $\tau(EX-SOP : f)$ , we have to choose  $g$  and  $h$  in different ways so that they satisfy (2.1). Thus, we have  $\tau(EX-SOP : f) = \min\{\tau(SOP : g) + \tau(SOP : h)\}$ .

**Definition 2.3:**  $\tau(EX-SOP : F)$  denotes the number of products in an EX-SOP  $F$ .

Logic functions can be grouped into classes by using simple transformations.

**Definition 2.4:** The set of functions which are identical under (a) the permutation of the variables and/or (b) the complementation (negation) of one or more variables are called NP-equivalent functions [6, 7, 9].  $f \stackrel{NP}{\sim} g$  denotes:  $f$  and  $g$  are NP-equivalent, and  $f \not\stackrel{NP}{\sim} g$  denotes:  $f$  and  $g$  are not NP-equivalent.

For the NP-equivalent functions we have the following:

**Property 2.1:** If  $f \stackrel{NP}{\sim} g$ , then  $\tau(SOP : f) = \tau(SOP : g)$  and  $\tau(EX-SOP : f) = \tau(EX-SOP : g)$ .

### 2.1. Modified Coordinate Representation and $\mu$ -Equivalence Classes

In the following, we first show a new representation of logic functions, called *modified coordinate representation*, and introduce a novel equivalence relation, called  $\mu$ -equivalence of logic functions. We present then properties of modified coordinate representation and  $\mu$ -equivalence classes.

**Definition 2.5:**  $w(f)$  denotes the number of true minterms of the function  $f$ .

**Definition 2.6:** [6] The coordinate representation of  $f$ ,  $COR(f)$  of a five-variable function consists of 32 integers:  $COR(f) = (c_0; c_1, c_2, c_3, c_4, c_5; c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{24}, c_{25}, c_{34}, c_{35}, c_{45}; c_{123}, c_{124}, c_{125}, c_{134}, c_{135}, c_{145}, c_{234}, c_{235}, c_{245}, c_{345}; c_{1234}, c_{1235}, c_{1245}, c_{1345}, c_{2345}; c_{12345})$ .

And  $c$ 's are calculated as follows:

$$c_0 = 2^{n-1} - w(f),$$

$$c_i = 2^{n-1} - w(f \oplus x_i), \quad i \in L;$$

$$c_{ij} = 2^{n-1} - w(f \oplus x_i \oplus x_j), \quad i, j \in L;$$

$$c_{ijk} = 2^{n-1} - w(f \oplus x_i \oplus x_j \oplus x_k), \quad i, j, k \in L;$$

$$c_{ijkl} = 2^{n-1} - w(f \oplus x_i \oplus x_j \oplus x_k \oplus x_\ell), \quad i, j, k, \ell \in L;$$

$$c_{12345} = 2^{n-1} - w(f \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5),$$

where  $n = 5$  and  $L = \{1, 2, 3, 4, 5\}$ . The elements of  $COR(f)$  which are separated by ';' (semicolon) form a group.

**Definition 2.7:** The modified coordinate representation of  $f$ ,  $\mu(f)$  of a five-variable function consists of 32 integers:  $\mu(f) = (d_0; d_1, d_2, d_3, d_4, d_5; d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45}; d_{123}, d_{124}, d_{125}, d_{134}, d_{135}, d_{145}, d_{234}, d_{235}, d_{245}, d_{345}; d_{1234}, d_{1235}, d_{1245}, d_{1345}, d_{2345}; d_{12345})$ .

And is calculated from  $COR(f)$  as follows: (a)  $d_0 = c_0$ ; (b)  $d_i$  ( $i \in \{1, 2, 3, 4, 5\}$ ) are obtained from  $c_i$  by deleting the sign and then rearranging the elements of the group in ascending order;  $d_{ij}$ ,  $d_{ijk}$  and  $d_{ijkl}$  ( $i, j, k, \ell \in \{1, 2, 3, 4, 5\}$ ) are obtained in similar ways; (c)  $d_{12345}$  is obtained by deleting the sign of  $c_{12345}$ .

**Theorem 2.1:**  $\mu(f)$  is invariant under (a) the permutation of the variables and/or (b) the complementation of one or more variables.

**Definition 2.8:** Two functions  $f$  and  $g$  are  $\mu$ -equivalent, denoted by  $f \stackrel{\mu}{\sim} g$ , if and only if, they have the same modified coordinate representation.

From the definitions of the NP-equivalent and  $\mu$ -equivalent functions, and by Theorem 2.1, we have the following:

**Property 2.2:** If  $f \stackrel{NP}{\sim} g$ , then  $f \stackrel{\mu}{\sim} g$ .

Note that the converse of Property 2.2 is not always true.

**Observation 2.1:** Among the five-variable functions, there exist functions  $f$  and  $g$  such that  $f \stackrel{\mu}{\sim} g$  but  $f \not\stackrel{NP}{\sim} g$ .

<sup>1</sup>The proof of the theorems can be found from the authors.

### III. MINIMIZATION OF EX-SOPS

In this section, we develop an algorithm to minimize EX-SOPs, where the two SOPs of the EX-SOP cannot share products.

#### 3.1. Idea for Minimization

The following theorem is the basis of the minimization of EX-SOPs.

**Theorem 3.1:** *Let  $f$  be an  $n$ -variable function and  $\mathcal{G}_n$  be the set of all the  $n$ -variable functions. Then,*

$$\tau(\text{EX-SOP}: f) = \min_{g \in \mathcal{G}_n} \left\{ \tau(\text{SOP}: g) + \tau(\text{SOP}: f \oplus g) \right\}. \quad (3.1)$$

#### 3.2. Reduction of Search Space

Theorem 3.1 shows that for the given  $n$ -variable function  $f$ , we have to check  $2^{2^n}$  different  $g$ 's, and choose  $g$  that produce a minimum value for  $\tau(\text{SOP} : g) + \tau(\text{SOP} : f \oplus g)$ . This search space is very large, even for  $n = 5$ . The following lemma shows that we can drastically reduce this search space.

**Theorem 3.2:** *In Theorem 3.1, suppose we need to find an EX-SOP with fewer than  $t$  products. If we consider  $g$ 's so that  $\tau(\text{SOP} : g)$  are in increasing order, then we have only to consider those  $g$ 's, such that*

$$\tau(\text{SOP} : g) \leq \lceil t/2 - 1 \rceil,$$

where  $\lceil k \rceil$  denotes the least integer greater than or equal to  $k$ .

#### 3.3. Minimization Algorithm: Straightforward

Based on Theorem 3.1, the following is a straightforward algorithm to minimize EX-SOPs:

**Algorithm 3.1:** (EX-SOP minimization: Straightforward)

1. Let  $f$  be the  $n$ -variable function to be represented as an EX-SOP, and  $\mathcal{G}_n$  be the set of all the  $n$ -variable functions.  $\mathcal{G}_n$  is arranged in ascending order of  $\tau(\text{SOP} : g)$ , where  $g \in \mathcal{G}_n$ .
2. **best** shows the minimum number of products in EX-SOPs ever found. **sol** shows a pair of  $n$ -variable functions.  
 $\text{best} \leftarrow \tau(\text{SOP} : f); \quad \text{sol} \leftarrow (f, 0);$
3. For each  $g \in \mathcal{G}_n$  (sequentially from the beginning of  $\mathcal{G}_n$ ) such that  $\tau(\text{SOP} : g) \leq \lceil \text{best}/2 - 1 \rceil$  **do**  
 $\text{temp} \leftarrow \tau(\text{SOP} : g) + \tau(\text{SOP} : f \oplus g);$   
**if**  $\text{temp} < \text{best}$  **then**  
 $\text{best} \leftarrow \text{temp}; \quad \text{sol} \leftarrow (g, f \oplus g);$   
**endif**  
**repeat**
4. Return **best** and **sol**.

For up to four-variable functions, Algorithm 3.1 produces solutions very quickly. However, for five-variable functions, it is rather time consuming. The computation techniques for the five-variable functions are considered in the next section.

### IV. STRATEGY FOR FIVE-VARIABLE EX-SOP MINIMIZATION

In this section, we first develop several techniques to reduce the computation time for five-variable functions. We then present a new algorithm to minimize EX-SOPs with five variables. The most time consuming part of Algorithm 3.1 is the computation of  $\tau(\text{SOP} : g) + \tau(\text{SOP} : f \oplus g)$  in step 3. The techniques we used to obtain  $\tau(\text{SOP} : g)$  and  $\tau(\text{SOP} : f \oplus g)$  are different.

#### 4.1. Obtaining $\tau(\text{SOP} : g)$ and Elimination of Redundant Computation

We can quickly obtain  $\tau(\text{SOP} : g)$  in Algorithm 3.1, by using a table of  $g \in \mathcal{G}_n$  and the corresponding  $\tau(\text{SOP} : g)$ . According to Theorem 3.2, we can reduce the number of  $g$ 's for Theorem 3.1 by considering them ( $g$ 's) such that  $\tau(\text{SOP} : g)$  is in ascending order. We found that for five-variable functions, maximum value of  $\tau(\text{SOP} : g)$  is four. For five-variable, the number of  $g$ 's such that  $\tau(\text{SOP} : g) \leq 4$  is 16,888,780, and it is inconvenient to work with a table of this size. To reduce the table size, we use NP-equivalence classes. From Property 2.1, the number of products in minimum SOPs for the NP-equivalent functions are equal. Every NP-equivalence class has an NP-representative function. There are only 6,138 NP-representative functions whose minimum SOPs require up to four products. Thus, we use the *sorted function table* shown in Fig. 2. The left column of the sorted function table stores only those NP-representative function  $g_{rep}$ 's, such that  $\tau(\text{SOP} : g_{rep}) \leq 4$ , and the right column stores the corresponding  $\tau(\text{SOP} : g_{rep})$ . The data in this table is arranged in ascending order of  $\tau(\text{SOP} : g_{rep})$ . We access the sorted function table sequentially from the beginning to get an NP-representative function  $g_{rep}$  and the corresponding  $\tau(\text{SOP} : g_{rep})$ . We then obtain  $g$ 's by generating all the function of the class  $g_{rep}$ . Since,  $\tau(\text{SOP} : g_{rep})$  and  $\tau(\text{SOP} : g)$  are equal, we obtain  $\tau(\text{SOP} : g)$  without much effort.

#### 4.2. Computation of $\tau(\text{SOP} : f \oplus g)$ : The Time Consuming Part

We have shown in Section 4.1 that  $\tau(\text{SOP} : g)$  can be quickly obtained from the sorted function table. Thus, the most time consuming part of Algorithm 3.1 is the

32 bits	
6,138 entries	$g_{rep}$
	$\tau(SOP : g_{rep})$

Fig. 2. NP-representative functions with  $\tau(SOP : g_{rep}) \leq 4$ .

32 bits	
1,228,158 entries	$h_{rep}$
	$\tau(SOP : h_{rep})$

Fig. 3. NP-representative functions.

32 integers		
149,466 entries	$\mu(h)$	$t_{low}(h)$
	$t_{up}(h)$	

Fig. 4.  $\mu$ -representative functions.

computation of  $\tau(SOP : f \oplus g)$ . A straightforward computation of  $\tau(SOP : f \oplus g)$  is time consuming. Thus, instead of doing logic minimization, we use a table of all the five-variable functions  $h$  and the corresponding  $\tau(SOP : h)$ . But, the total number of five-variable functions is  $2^{32} \approx 4.3 \times 10^9$ , and it is impractical to store a table of this size.

#### 4.2.1. Reduction of Table Size: Use of NP-Equivalence Classes

To reduce the table size, we use a cost table for all the NP-representative functions of five variables. The number of NP-equivalence classes of five-variable functions is 1,228,158. Fig. 3 shows the *cost table*. The left column of the cost table stores all the NP-representative functions  $h_{rep}$  and the right column stores the corresponding  $\tau(SOP : h_{rep})$ . We arranged the data in the left column of the cost table, which helps to quickly locate the position of an NP-representative function in the table. Note that the data in the sorted function table is a subset of the data in the cost table. Also the arrangements of data in the two tables are different. For a given function  $h_{given}$ , to obtain  $\tau(SOP : h_{given})$  from the cost table, first we compute the NP-representative function  $h_{rep}$  of  $h_{given}$ . Since the number of products is invariant under the NP-equivalence class, we have  $\tau(SOP : h_{given}) = \tau(SOP : h_{rep})$ . Using  $h_{rep}$  for the cost table look-up, we obtain  $\tau(SOP : h_{given})$ . However, to get the NP-representative function  $h_{rep}$  from a given function  $h_{given}$  is still rather time consuming.

#### 4.2.2. Quick Estimation of $\tau(SOP : f \oplus g)$ : Use of $\mu$ -Equivalence Classes

To speed-up the computation, we use  $\mu$ -equivalence classes, which we have already introduced in Section 2.1.  $\mu(h)$ , the modified coordinate representation of  $h$ , is quickly calculated from  $h$  and have Property 2.2. But, Observation 2.1 shows that  $\mu(h)$  corresponds to more than one NP-equivalence classes for some  $h$ . All the 1,228,158

NP-equivalence classes of five-variable functions can be partitioned into 149,466  $\mu$ -equivalence classes. Although  $\mu$ -equivalence classes cannot uniquely identify the NP-equivalence classes, we can use them to quickly estimate the value of  $\tau(SOP : f \oplus g)$  in step 3 of Algorithm 3.1. Thus, we use the *modified cost table*, which is shown in Fig. 4. The left column of the modified cost table stores the distinct values of  $\mu(h)$  for all the five-variable function  $h$ . The middle and the right columns store the corresponding values of  $t_{low}(h) = \min_h \mu_{h_j} \{\tau(SOP : h_j)\}$  and  $t_{up}(h) = \max_h \mu_{h_j} \{\tau(SOP : h_j)\}$ , respectively.

Usually the differences between  $t_{low}(h)$  and  $t_{up}(h)$  are small. Thus, without computing  $\tau(SOP : f \oplus g)$  and using the modified cost table, very often we can show that  $\tau(SOP : g) + t_{low}(f \oplus g) \geq \text{best}$ , i.e.,  $\text{temp} \geq \text{best}$  in step 3 of Algorithm 3.1. This implies that using the modified cost table, very often we can avoid the computation of  $\tau(SOP : f \oplus g)$ . If we have  $\tau(SOP : g) + t_{low}(f \oplus g) < \text{best}$ , i.e., a possibility that  $\text{temp} < \text{best}$  in step 3 of Algorithm 3.1, only then we compute  $\tau(SOP : f \oplus g)$ . In many cases  $t_{low}(f \oplus g)$  and  $t_{up}(f \oplus g)$  are equal, and we get  $\tau(SOP : f \oplus g)$  from the modified cost table. When  $t_{low}(f \oplus g)$  and  $t_{up}(f \oplus g)$  are unequal, we obtain  $\tau(SOP : f \oplus g)$  from the cost table by using more time consuming routine. Since  $\mu(h)$  is an array of 32 integers, we use hash technique to look-up modified cost table.

#### 4.3. Algorithm for Five-Variable EX-SOP Minimization

Based on the above discussions and Theorem 3.1, an algorithm for the minimization of EX-SOPs with five variables is presented in the following:

**Algorithm 4.1:** (Minimization of EX-SOPs for five-variable)

0.  $f$ ,  $g$ ,  $h$  and  $g_{rep}$  represent functions, and  $t_g$ ,  $t_{exsop}$ ,  $t_{bound}$ ,  $t_{low}$ ,  $t_{up}$  and  $t_h$  represent the number of products. Read the cost table, the modified cost table and the sorted function table. Read the function to be minimized. Let it be  $f$ .

1.  $t_{\text{exsop}}$  denotes the minimum number of products in EX-SOP ever found. Let  $t_{\text{exsop}} \leftarrow 10$ .  $t_{\text{bound}}$  represents the upper bound on the number of products in the SOP for  $g$ . Let  $t_{\text{bound}} \leftarrow 4$ .
2. From the sorted function table: (a) take the first NP-representative function  $g_{\text{rep}}$  and (b) obtain  $t_g \leftarrow \tau(\text{SOP} : g_{\text{rep}})$ .
3. Generate all the functions of the NP-equivalence class  $g_{\text{rep}}$ . Take the first function of this class. Let the function be  $g$ .
4.  $h \leftarrow f \oplus g$ .
5. Calculate  $\mu(h)$ . By using  $\mu(h)$ , obtain  $t_{\text{low}} \leftarrow t_{\text{low}}(h)$  and  $t_{\text{up}} \leftarrow t_{\text{up}}(h)$  from the modified cost table.
6. If  $t_g + t_{\text{low}} \geq t_{\text{exsop}}$  (reduction of  $t_{\text{exsop}}$  is impossible using the current  $h$ ), then go to step 11.
7. If  $t_{\text{low}} = t_{\text{up}}$  ( $\tau(\text{SOP} : h)$  is obtained from the modified cost table), then  $t_h \leftarrow t_{\text{low}}$  and go to step 9.
8. Obtain  $t_h \leftarrow \tau(\text{SOP} : h)$  from the cost table.
9. If  $t_g + t_h \geq t_{\text{exsop}}$  (reduction of  $t_{\text{exsop}}$  is impossible using the current  $h$ ), then go to step 11.
10. (New solution is found.)  $t_{\text{exsop}} \leftarrow t_g + t_h$ . Save  $g$  and  $h$  as the latest solution.  $t_{\text{bound}} \leftarrow \lceil t_{\text{exsop}}/2 \rceil - 1$ . If  $t_{\text{bound}} \leq t_g$ , then go to step 12.
11. Take the next function  $g$  in the class  $g_{\text{rep}}$  (computed in step 3), and go to step 4. If there is no remaining function in this class, then from the sorted function table: (a) take the next NP-representative function  $g_{\text{rep}}$  and (b) obtain  $t_g \leftarrow \tau(\text{SOP} : g_{\text{rep}})$ . If  $t_{\text{bound}} < t_g$ , then go to step 12, otherwise go to step 3.
12. Print the latest solution saved in step 10, and  $t_{\text{exsop}}$  as the final number of products.

The execution time of Algorithm 4.1 mainly depends on  $t_{\text{bound}}$ , the upper bound on the number of products in the SOP for  $g$  of Theorem 3.1. In Algorithm 4.1,  $t_{\text{bound}}$  is defined in step 1 and it is updated in step 10.

#### V. SIMPLIFICATION OF EX-SOPS WITH SIX OR MORE VARIABLES

A simplified EX-SOP for the  $n$ -variable function can be obtained by using a pair of EX-SOPs for the  $(n - 1)$ -variable functions. The idea for this simplification is shown in the following.

**Theorem 5.1:** [3] Let  $\tau(\text{EX-SOP} : n)$  denote the maximum number of products required to realize an arbitrary  $n$ -variable function by a minimum EX-SOP. Then

$$\tau(\text{EX-SOP} : n) \leq 2\tau(\text{EX-SOP} : n - 1).$$

TABLE I  
NUMBERS OF FIVE-VARIABLE FUNCTIONS  
REQUIRING  $t$  PRODUCTS

$t$	SOP	ESOP	EX-SOP
0	1	1	1
1	243	243	243
2	20676	24948	25988
3	818080	1351836	1511996
4	16049780	39365190	47838990
5	154729080	545193342	694830748
6	698983656	2398267764	2678055614
7	1397400512	1299295404	870943300
8	1254064246	11460744	1760384
9	571481516	7824	32
10	160200992		
11	34140992		
12	6160176		
13	827120		
14	84800		
15	5312		
16	114		
av	7.46	6.16	6.02

av : average

Based on the idea presented in this section, a heuristic algorithm to simplify EX-SOPs with six or more variables has been developed [1]. The algorithm uses a table of minimum EX-SOPs for the representative functions of NP-equivalence classes of five variables.

#### VI. EXPERIMENTAL RESULTS

Using our EX-SOP minimization program, we minimized all the 1,228,158 representative functions of NP-equivalence classes of five variables. A five-variable function, on the average, took 9.54 CPU seconds on a DEC ALPHAStation 200. This average is obtained by minimizing 10,000 randomly generated functions with 16 true minterms. For five-variable functions, when the number of products in the minimum EX-SOP is 6 and 9 (worst case), we could minimize each function within 3 and 38 CPU seconds, respectively, on the same machine.

Table I shows the numbers of five-variable functions requiring  $t$  products by different minimum expressions.<sup>1</sup> In this table, data for SOPs and ESOPs are taken from [13]. EX-SOPs were minimized by Algorithm 4.1. For five-variable functions, on the average, minimum EX-SOPs require 6.02 products while minimum SOPs require 7.46 products. For the five-variable functions, on the average, minimum EX-SOPs require fewer products than minimum ESOPs. We found that for four and five-variable

<sup>1</sup>In Table I,  $\text{av} = \left( \sum_t (t \times \text{number of functions requiring } t \text{ products}) \right) / \text{total number of functions}$ . Total number of five-variable functions is  $2^{32}$ .

functions, the upper bounds on the number of products in minimum EX-SOPs (when two SOPs cannot share products) are 5 and 9, respectively. Thus, from Theorem 5.1, minimum EX-SOPs with  $n$  variables require at most  $9 \cdot 2^{n-5}$  ( $n \geq 6$ ) products. For five-variable functions, there is only one NP-equivalence class whose minimum EX-SOP requires 9 products. The NP-representative function of this class is 177e7ee9<sub>16</sub>. In this class, only 32 functions are equivalent.

## VII. CONCLUSIONS AND COMMENTS

In this paper, we presented minimization algorithms for AND-OR-EXOR three-level networks (EX-SOPs) for up to five variables, where two SOPs of the EX-SOP cannot share products. We developed the concept of  $\mu$ -equivalence of logic functions and used it to reduce the computation time of the minimization program. We minimized all the 1,228,158 representative functions of NP-equivalence classes of five variables. We have completed the table of minimum EX-SOPs with up to five variables and showed that minimum EX-SOPs for five-variable functions require up to 9 products. For  $n$ -variable functions, the upper bound on the number of products in minimum EX-SOPs is at most  $9 \cdot 2^{n-5}$  ( $n \geq 6$ ). This is tighter than the previously known one:  $5 \cdot 2^{n-4}$  ( $n \geq 4$ ) [3]. This upper bound is smaller than  $2^{n-1}$ , the upper bound for the minimum SOPs. We found that for five-variable functions, on the average, minimum EX-SOPs require 6.02 products while minimum SOPs require 7.46 products. In our minimization algorithms, we did not consider the sharing of products between two SOPs of an EX-SOP. If we consider sharing of products, we can realize EX-SOPs with fewer products for many functions. We consider this problem in a separate paper [2], where the tables of minimum EX-SOPs with up to five variables are used. The table of minimum EX-SOPs with five variables is also used in the heuristic simplification program for EX-SOPs with six or more inputs [1].

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