Abstract—We study coverage and outage in large networks containing two kinds of fixed wireless transceivers that we call nodes and base stations (BSs) respectively. The nodes have common wireless capabilities, with the BSs having in addition direct wideband connections to the wired infrastructure. Nodes can communicate with the “outside world” only through the BSs. Connections to nodes without a direct (i.e., a single-hop) wireless connection to any BS are established through other nodes serving as wireless repeaters. The locations of the nodes are assumed randomly and uniformly distributed over the entire service area, or clustered following a clumped Poisson process. The BSs are also assumed randomly and uniformly distributed over the given service area. We evaluate the probability of a potential node to have a “working” wireless connection to any of the BSs within a fixed but arbitrary number of hops, as a function of the densities of the BSs and nodes and the parameters of the wireless links, accounting for both fast and shadow fading. We provide exact expressions for the outage probability of an node, and lower bounds when evaluation of the exact expression is impractical. Our results allow simple comparisons with other means of extending the BSs’ reach, thereby allowing network designers to choose the optimal solution.

Keywords: mesh networking, multi-hop, outage probability

I. INTRODUCTION

We study a large network of fixed wireless transceivers that we call nodes. These nodes have certain common wireless capabilities and some, called base stations (BSs), also have direct wideband connections to the wired infrastructure. Nodes can communicate with the “outside world” only through the BSs. To augment connectivity, connections to nodes without a direct (i.e., a single-hop) wireless connection to any BS are established through other nodes serving as wireless repeaters, as long as the number of hops does not exceed a prescribed limit. The BSs are assumed randomly and uniformly distributed over the entire service area, and their locations cannot be predicted ahead of time; we therefore assume that their locations are random. We further assume that due to practical constraints, availability of high speed wired connections and economic considerations, BSs are sparse, and often cannot be positioned based on coverage considerations only. This scenario applies in particular to service providers offering wideband wireless connectivity in an area where high speed optical cables are scarce and owned by different entities. To account for this reality, we assume that the BSs, like the regular nodes, are also placed randomly over the service area, recognizing that this is a “worst case” scenario. The focus of this paper is the $t$-hop outage probability, defined as the probability that a node (e.g., a sensor) cannot connect to any of the BSs in $\leq t$ hops, evaluated as a function of the densities of the nodes and the statistics of fast and shadow fades on the wireless links. Since $t$-hop outage is equivalent to the minimum number of hops from the node to any BS being greater than $t$, we see that the collection of $t$-hop outage probabilities for all $t$ is equivalent to the complementary cumulative distribution function (cdf) of the minimum number of hops required for a packet to get from an arbitrary node to a BS.

This paper is organized as follows: following a review of related work, we first state and prove for our reference a well-known result that will be extensively used later. We then describe the location models to be used and the wireless channel model, and compute in closed form the exact probability that an arbitrary node is isolated from BSs and other nodes when the node locations are either homogeneous or clump Poisson points. We then derive bounds on the general $t$-hop outage probability for both node location models using two different approximations. These require the distribution of the distance between two nodes conditioned on their being connected, which is derived next. We compare our analytical results with simulations and end by summarizing our conclusions.

II. RELATED WORK

A. Complete Connectivity of a Network of Nodes

The issue of connectivity of wireless ad-hoc networks has attracted the attention of many researchers. In this kind of network the interesting question is the probability that the network of nodes forms a connected graph, so that a route can be found from any node to any other node.

For no fading and nodes with fixed range $r_n$ distributed over a $d$-dimensional space $\mathcal{S} = [0, l]^d$ according to a homogeneous Poisson point process with intensity $\lambda_n$, several researchers investigated the question of whether the collection of all nodes in a given finite region forms a fully-connected network. In [1], Philips et al. prove that for $d = 2$ and a given $\lambda_n$, the asymptotic (as $l \to \infty$) probability that all points in $\mathcal{S}$ are within distance $r_n$ from at least one node is 1 or 0 depending on whether $r_n \geq \sqrt{\ln(\text{area}(\mathcal{S}))/(\pi \lambda_n)}$, and that in the latter case the probability that the network is fully connected is also 0. Santi and Blough [2] study the case where the number of nodes in $\mathcal{S}$ is fixed at some $n$. For $d = 1$, they prove that asymptotically in $l$, the network is connected or not connected with high probability (WHP) depending on whether $nr_\epsilon \geq 2l\ln l$ or $\leq (1-\epsilon)l\ln l$ for some $\epsilon \in (0,1)$. They derive necessary and sufficient conditions for connectedness WHP for $d = 1, 2, 3$ and stationary nodes, and also present simulation results of mobility scenarios showing that for a large range of parameters, mobile networks are effectively stationary as...
far as connectivity is concerned. For \( d = 2, 3 \) they further investigate through simulations the minimal values of \( r_n \) and \( n \) that ensure either a connected graph, or the formation of a single connected component that includes a large fraction (e.g., 90%) of the nodes.

### B. Connectivity of a Single Node to Other Nodes

A related issue is that of coverage: can a wireless terminal moving randomly throughout the service area maintain continuous (multihop, if necessary) connectivity with a network of fixed wireless nodes?

The literature on multihop outage probability calculations usually assumes no fading. Bettstetter and Eberspaecher [3] investigate the probability distribution of the minimal number of hops between a randomly-selected source and destination node, where all \( n \) nodes are uniformly distributed on a rectangular area of size \( A = a \times b, b \leq a \). All the nodes have the same fixed transmission range \( r_n \). They derive the probability that two randomly selected nodes on the square can communicate between themselves directly, i.e., are 1-hop connected as a function of \( a, b \), and \( r_n \). They then derive the probability that two nodes on the square are 2-hop connected, but not 1-hop connected, i.e., that the two nodes are further than \( r_n \) apart, but at least one other node exists that is within \( r_n \) from both nodes. Ignoring border effects (at the edges of the square), they provide the precise expression of the probability of that event. Bettstetter and Eberspaecher then provide simulation results for some larger number of hops and analytic results for the case where the node density \( n/A \) increases without limit. Assuming nodes distributed in the plane according to a two-dimensional spatial Gaussian distribution around the origin, Miller [4] provides an analytical approximation, as well as a curve-fitting function derived by regression to simulated data, for the probability of a 2-hop connection between two randomly selected nodes. He also provides an upper bound to the probability of a 1-hop connection. Chandler [5] considers a network with node locations given by points of a homogeneous Poisson process and derives an expression for the t-hop outage probability. Though not stated explicitly in his paper, his derivation relies on some “independence assumptions” (discussed later in the present paper, and in [6]) which imply that for \( t \geq 3 \), his expressions are actually lower bounds on the exact t-hop outage probabilities.

For the case of fading, and nodes distributed according to a planar homogeneous Poisson process, Bettstetter and Hartmann [7] derive an expression for the node isolation probability (single-hop outage), but do not obtain a closed form for this expression. Orriss and Barton [8] derive an exact closed-form expression for the node isolation probability with fading for arbitrary densities. They also allow two-slope propagation models and varying spatial densities. (In this paper we provide a different and independent derivation of an exact closed-form expression [13] for the single-hop outage probability, which matches the one in [8].)

### C. Connectivity in Hybrid Ad-Hoc Networks

The above studies focused on pure ad hoc networks. To boost connectivity, some researchers have recently examined “hybrid networks” [9] [10], where a sparse network of BSs is placed on a regular grid within an ad hoc network. The BSs are all connected via a high data-rate wired infrastructure and serve as relays for data packets generated by the nodes of the ad hoc network. Dousse et al. [9] study hybrid networks where BSs positioned on a fixed square grid are added to a low density ad-hoc network with nodes placed according to a homogeneous Poisson point process. They take the transmission range \( R \) to be fixed and allow the intensity of the Poisson process to change. Like [1] and [2], they also allow any number of hops, and provide simulation results of the probability of having a disconnected node. As above, their objective is to allow connectivity between any two arbitrarily chosen nodes, whether through a BS, or not.

### III. Central Result

We begin by re-stating a well-known result in a form that is most appropriate for our purposes. For completeness, we also include the proof of this result, which illustrates several techniques that will be used subsequently in the outage probability calculations.

**Theorem 1:** Let the number of objects \( N \) in a given region be a Poisson random variable with mean \( \mu \). Suppose that conditioned on \( N = n \), the events \( E_i \) \((i = 1, \ldots, n)\) that the \( i \)th object has a desired property are independent and have the same probability of occurrence \( p = P(E_i|N = n) \) for all \( n > 0 \). Then \( N' \), the number of objects (out of these \( N \)) that have the desired property, is a Poisson random variable with mean \( \mu p \).

**Proof:** With the usual convention that an empty sum is zero and an empty product is unity, we can write

\[
N' = \sum_{i=1}^{N} I_i,
\]

where \( \{I_i\}_{i=1}^{\infty} \) are the indicator functions of the events that these objects have the desired property. The moment generating function (mgf) of \( N' \) is thus

\[
\phi_{N'}(s) = \mathbb{E}[s^{N'}] = \mathbb{E}\left\{\mathbb{E}\left[s^{N'}|N\right]\right\} = \sum_{n=0}^{\infty} \mathbb{P}(N = n) \mathbb{E}\left[\prod_{i=1}^{n} s^{I_i} | N = n\right]. \quad (1)
\]

Now, conditioned on \( N = n, I_1, \ldots, I_n \) are i.i.d. Bin(1, \( p \)), so

\[
\mathbb{E}\left[\prod_{i=1}^{n} s^{I_i} | N = n\right] = \prod_{i=1}^{n} \mathbb{E}[s^{I_i} | N = n] = \left\{\mathbb{E}[s^{I_i} | N = n]\right\}^{n} = [1 - p(1-s)]^{n}, \quad (2)
\]
where we use the known mgf of the Bin$(1, p)$ distribution. Substituting (2) into (1) and using the known form of the mgf of the Poiss$(\mu)$ distribution yields

\[
\phi_N'(s) = \sum_{n=0}^{\infty} \mathbb{P}\{N = n\} [1 - p(1 - s)]^n = \phi_N(1 - p(1 - s)) = \exp\{-\mu\pi [1 - (1 - p(1 - s))]\} = \exp[-\mu p(1 - s)],
\]

which is the mgf of the Poiss$(\mu p)$ distribution. In other words,

\[
N' \sim \text{Poiss}(\mu p).
\]

In particular, the probability that none of the objects in this region has the desired property is

\[
\mathbb{P}\{N' = 0\} = \exp(-\mu p).
\]

We now discuss how this can be used to derive some results on outage probabilities as follows.

IV. LOCATION MODEL FOR BSSs

Define the disk with radius $r$ centered at $(x, y)$ by

\[
B(x, y; r) = \{(x', y') : (x' - x)^2 + (y' - y)^2 \leq r^2\},
\]

and the punctured disk obtained by deleting the center:

\[
B'(x, y; r) = B(x, y; r) \setminus \{(x, y)\}.
\]

We begin by assuming that BSs are points of a homogeneous Poisson process on the plane with intensity $\lambda_{BS}$:

(i) The number of BSs in any finite region is a Poisson random variable with mean given by $\lambda_{BS} \times \text{area of the region}$;

(ii) The numbers of BSs in two disjoint finite regions are independent; and

(iii) Conditioned on a given number of BSs in a chosen region, the locations of these BSs are independently and uniformly distributed over that region.

Define $N_{BS}(A) = \#$ of BSs in region $A$. Then

\[
N_{BS}(A) \sim \text{Poiss}(\lambda_{BS} \times \text{area}(A)),
\]

\[
\mathbb{P}\{N_{BS}(A) = n\} = e^{-\lambda_{BS} \times \text{area}(A)} \left[ \frac{\lambda_{BS} \times \text{area}(A)}{n!} \right]^n, \quad n \geq 0.
\]

V. PROBABILITY THAT A GIVEN NODE CANNOT CONNECT DIRECTLY TO ANY BS

Let the region of interest be $B(0, 0; r_0)$, the disk of radius $r_0$ centered at the origin. Let the objects be the BSs, which are points of a homogeneous planar Poisson process with intensity $\lambda_{BS}$, so that $N_{BS}(r_0)$, the number of BSs in $B(0, 0; r_0)$, is Poiss$(\lambda_{BS} \pi r_0^2)$. Let the desired property for each BS in this region be the existence of a connection, i.e., a ‘working’ link between that BS and the point $(0, 0)$, an event with probability $p(r_0)$, say. Then from (5), the probability that a node at $(0, 0)$ has no connection to any BS in the region is $\exp[-\lambda_{BS} \pi r_0^2 p(r_0)]$. Finally, the probability that a node at $(0, 0)$ has no connection to any BS in the plane follows from the monotone convergence of probability measures from above as we let $r_0 \to \infty$:

\[
q_{BS} = \lim_{r_0 \to \infty} e^{-\lambda_{BS} \pi r_0^2 p(r_0)} = \exp \left[ -\lambda_{BS} \pi \lim_{r_0 \to \infty} r_0^2 p(r_0) \right].
\]

A. Radio Propagation Model

We assume the following:

1. Attenuation with distance, shadow and fast fading.

2. Shadow fades on all links are i.i.d., and similarly the fast fades on all links are also i.i.d. and independent of all shadow fades.

3. We assume that if transmissions from a node (always with power $P_T$) can reach a BS, then transmissions from this BS can reach the node. Node $i$ is said to have a connection to (i.e., is one hop away from, or has a ‘working’ link to) a given BS if and only if the received power at the BS exceeds some given threshold $P_{\text{min}}$ (assumed the same at all nodes):

\[
G X^2 \frac{P_T}{R^\delta} 10^{Z/10} > P_{\text{min}}
\]

\[
\Leftrightarrow \left( \frac{R}{r_{BS}} \right)^\delta < X^2 10^{Z/10}, \quad r_{BS} \equiv \left( \frac{G P_T}{P_{\text{min}}} \right)^{1/\delta},
\]

where $\delta$ is the distance-loss exponent, $X$ is the (normalized) envelope coefficient of fast fading (so that $\mathbb{E}[X^2] = 1$), $10^{Z/10}$ is the shadow fading, $R$ is the distance between the node and the BS, and $G$ is a constant, taking into account parameters like antenna gain, antenna height (again assumed equal for all nodes), etc. Observe that (7) reduces to $R < r_{BS}$ if $Z \equiv 0$, $X \equiv 1$, so $r_{BS}$ is the range of the BS in the absence of fading.

4. Shadow fading $10^{Z/10}$ is assumed log-normal, i.e., $Z \sim \mathcal{N}(0, \sigma^2)$, where $\sigma$ is the same for all links.

B. Calculation of $\lim_{r_0 \to \infty} r_0^2 p(r_0)$

We now calculate the probability $p(r_0)$ that there is a connection between $(0, 0)$ and an arbitrary BS in $B(0, 0; r_0)$. Let the distance of this BS from the origin be $R$. From (7), the event that there is a connection between the BS and the node is equivalent to

\[
R < r_{BS} X^2 / h Z / \delta, \quad h = \ln \frac{10}{10}.
\]

Since the BSs are points of a homogeneous planar Poisson process, we know that the given BS is distributed uniformly over $B(0, 0; r_0)$, so the probability density function (pdf) of $R$ is given by

\[
f_R(r) = \frac{2r}{r_0^3}, \quad 0 \leq r \leq r_0.
\]

Then we may evaluate the asymptotic probability of the event (8) as follows:

\[
\lim_{r_0 \to \infty} r_0^2 p(r_0) = \lim_{r_0 \to \infty} r_0^2 \mathbb{P}\{R < r_{BS} X^2 / h Z / \delta\}
\]
TABLE I

<table>
<thead>
<tr>
<th>Model</th>
<th>pdf $f_X(x)$</th>
<th>$\mathbb{E}[X^{4/\delta}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh</td>
<td>$2x \exp(-x^2)$</td>
<td>$\Gamma(1 + \frac{2}{\delta})$</td>
</tr>
<tr>
<td>Rice</td>
<td>$\frac{2\mu x \exp(K + (K + 1)x^2)}{\exp(K + (K + 1)x^2)} \times I_0(2x \sqrt{K(K + 1)})$</td>
<td>$\frac{eK \Gamma(1 + \frac{2}{\delta})}{(1 + K)^{2/\delta}} \times {}_1F_1(1 + \frac{2}{\delta}; 1; K)$</td>
</tr>
<tr>
<td>Nakagami</td>
<td>$\frac{2m^m x^{2m-1}}{\exp(mx^2) \Gamma(m)}$</td>
<td>$\frac{\Gamma(m + \frac{2}{\delta})}{m^{2/\delta} \Gamma(m)}$</td>
</tr>
</tbody>
</table>

TABLE II

Table of expressions for $\mathbb{E}[X^{4/\delta}]$.

\[
\begin{align*}
= & \int_0^{\infty} f_X(x) \int_0^{\infty} f_Z(z) \times \lim_{r_0 \to \infty} r_0^2 
\int_{\min\{r_0, r_{BS}x^2/\delta, h^2z/\delta\}}^{\infty} \frac{2r}{r_0^3} dr dz dx \\
= & r_{BS}^2 \int_0^{\infty} x^{4/\delta} f_X(x) dx \int_{-\infty}^{\infty} e^{2h^2z/\delta} f_Z(z) dz \\
= & r_{BS}^2 \mathbb{E}[X^{4/\delta}] \mathbb{E}[e^{2hZ/\delta}].
\end{align*}
\]

Thus from (6), the probability that a given node at $(0, 0)$ has no connection to any BS is given by

\[
q_n = \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \mathbb{E}[X^{4/\delta}] \mathbb{E}[e^{2hZ/\delta}] \right\}.
\]  

Since $Z \sim N(0, \sigma^2)$, we have

\[
\mathbb{E}[e^{2hZ/\delta}] = \exp(2h^2\sigma^2/\delta^2) = \exp(2\alpha^2),
\]

where

\[
\alpha = \frac{h\sigma}{\delta} = \frac{\sigma \ln 10}{10\delta}.
\]

Then we have, from (10),

\[
q_n = \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \exp(\alpha^2)^2 \mathbb{E}[X^{4/\delta}] \right\}.
\]

VI. Probability that a node cannot connect directly to any other node

A. Nodes are points of a homogeneous planar Poisson process

If the nodes are points of a homogeneous planar Poisson process with intensity $\lambda_n$, each node has range $r_n$ in the absence of fading, and the node-node channel model is the same as the node-BS channel model (possibly with a different value for $G$), then it is clear that the probability that a node cannot connect directly to any other node in the entire plane is given by (12) with $\lambda_{BS}$ and $r_{BS}$ replaced by $\lambda_n$ and $r_n$ respectively:

\[
q_n = \exp \left\{ -\lambda_n \pi r_n^2 \exp(\alpha^2)^2 \mathbb{E}[X^{4/\delta}] \right\}.
\]

B. Nodes are points of a clumped Poisson process

It is common practice to study various aspects of wireless systems under the assumption that a known number of nodes are placed randomly and uniformly over the serviced area. Alternatively, if the service area is very large, possibly with the total number of nodes not known, an isotropic Poisson point process is used, where only the density of nodes per unit area is required. In practice, though, nodes are often highly concentrated in certain regions, like office building, or shopping malls, for instance, while other regions are almost vacant. It is intuitively clear that the results we are presenting in this paper might turn out to be significantly biased in such cases. To address non-uniform distribution scenarios, we have introduced the clumped Poisson process in an earlier paper [12]. Note, however, that this applies only to the node locations, and that the BS locations continue to be points of a homogeneous Poisson process, as before.

1) The Clumped Poisson Model:

(a) The clump centers are points of a homogeneous Poisson process on the plane with intensity $\lambda_c$;

(b) The number of nodes in each clump is a Poisson random variable with mean $\mu_c$, and these nodes are i.i.d. uniformly distributed over a disk of radius $r_c$;

(c) The number and location of nodes is independent across clumps.

2) Probability that a node cannot connect to a clump:

We say that a given node has a connection to a clump if and only if there is at least one node in that clump to which the given node has a connection. Clearly, the probability that a given node at $(0, 0)$, say, cannot connect to any other node is just the probability that this node has no connection to any clump. Since the clump placement process is homogeneous planar Poisson, we see from argument yielding (6) that this probability is

\[
q_n = \lim_{r_0 \to \infty} \frac{\alpha}{c_\pi} \exp(-\lambda_c \pi \frac{r_0^2}{c} \frac{c}{c_\pi} \lim_{r_0 \to \infty} r_0^2 p_c(r_0)),
\]

where now $p_c(r_0)$ is now the probability that the node at $(0, 0)$ has a connection to a clump whose clump center is uniformly
distributed over $B(0, 0; r_0)$. Thus, we may write
\[
p_{c}(r_0) = 1 - \int_{0}^{r_0} \frac{2r}{\pi r_0^2} q_c(r) \, dr = \int_{0}^{r_0} \frac{2r}{\pi r_0^2} \left[ 1 - q_c(r) \right] \, dr,
\]
where $q_c(r)$ is the probability that the node at $(0, 0)$ has no connection to a clump whose center is located at a distance $r$ from $(0, 0)$.

Let us denote the number of nodes in a clump by $N_{c,n}$. We assume that conditioned on $N_{c,n} = k$ and the clump center being at a distance $r$ from $(0, 0)$, the events that the $k$ individual nodes in the clump have no connection to the node at $(0, 0)$ are independent and have the same probability. Now, we can apply Theorem 1 once more, taking the region to be the clump itself, our objects to be the nodes in the clump, and the desired property to be that of having a connection to the node at $(0, 0)$. Then we have
\[
q_c(r) = \exp[-\mu_c p_{c,n}(r)],
\]
where $p_{c,n}(r)$ is the probability that an arbitrary node in a clump at a distance $r$ from $(0, 0)$ has a connection to $(0, 0)$. We now focus on the derivation of $p_{c,n}(r)$. Without loss of generality, let the clump center be located at $(r, 0)$. Conditioned on $X = x$ and $Z = z$, the fast and shadow fades respectively from the node to $(0, 0)$ is clear that there is a connection between this node and $(0, 0)$ if and only if the node lies in the intersection of the clump region $B(r, 0; r_c)$ and the disk $B(0, 0; r_0x^{2/\delta}e^{hZ/\delta}, r_0)$, which represents the range of the node at $(0, 0)$ given these fade values. \([\text{Strictly speaking, these disks should be replaced by their intersections with the overall region } B(0, 0; r_0), \text{ but we ignore this detail by assuming that } r_0 \text{ is sufficiently large}].\)

This approximation becomes exact when we take the limit as $r_0$ goes to infinity. Since the nodes in a clump are uniformly distributed over the clump region, the probability of this event is therefore
\[
\frac{\text{area} \left[ B(0, 0; r_0x^{2/\delta}e^{hZ/\delta}) \cap B(r, 0; r_c) \right]}{\text{area}[B(r, 0; r_c)]} = \frac{g(r; r_0x^{2/\delta}e^{hZ/\delta}, r_c)}{\pi r_c^2},
\]
where $[13, \text{equation (11)}] g(r; a, b)$ is the area of intersection of two disks of radii $a$ and $b$ respectively whose centers are $r$ apart:
\[
g(r; a, b) = \text{area}(B(r, 0; a) \cap B(0, 0; b)) = \begin{cases} \pi \left( \min\{a, b\} \right)^2, & r < |a - b|, \\
\frac{a^2 \cos^{-1} \left( \frac{r^2 + a^2 - b^2}{2ra} \right)}{2a} + \frac{b^2 \cos^{-1} \left( \frac{r^2 + b^2 - a^2}{2b} \right)}{2b}, & 0 < r < |a - b|, \\
\frac{1}{2} \sqrt{(-r + a + b)(r + a - b)} \sqrt{(r - a + b)(r + a + b)}, & |a - b| \leq r < a + b, \\
0, & r \geq a + b. \end{cases}
\]

Integrating the right hand side of (17) with respect to the pdfs of $X$ and $Z$, we obtain the corresponding unconditional probability to be
\[
p_{c,n}(r) = \frac{\mathbb{E}_X, Z \left[ g(r; r_0X^{2/\delta}e^{hZ/\delta}, r_c) \right]}{\pi r_c^2} = \int_{0}^{\infty} f_X(x) \int_{-\infty}^{\infty} \exp \left[ -z^2/(2\sigma^2) \right] x B\left(0, 0; r_0x^{2/\delta}e^{hZ/\delta}, r_c\right) \, dz \, dx.
\]
Substituting (19) into (16), we obtain
\[
q_c(r) = \exp \left\{ -\frac{\mu_c}{\pi r_c^2} \mathbb{E}_X, Z \left[ g(r; r_0X^{2/\delta}e^{hZ/\delta}, r_c) \right] \right\}.
\]
Substituting (21) into (15), we obtain
\[
p_c(r_0) = \int_{0}^{r_0} \frac{2r}{\pi r_0^2} \times \left( 1 - \exp \left\{ -\frac{\mu_c}{\pi r_c^2} \mathbb{E}_X, Z \left[ g(r; r_0X^{2/\delta}e^{hZ/\delta}, r_c) \right] \right\} \right) \, dr.
\]
Finally, substituting (22) into (14), we obtain the probability that a given node cannot connect directly to any clump to be
\[
q_n = \exp \left\{ -\lambda_c \pi \int_{0}^{\infty} 2r \times \left( 1 - \exp \left\{ -\frac{\mu_c}{\pi r_c^2} \mathbb{E}_X, Z \left[ g(r; r_0X^{2/\delta}e^{hZ/\delta}, r_c) \right] \right\} \right) \, dr \right\}.
\]

VII. DISTRIBUTION OF NUMBER OF NEIGHBORS OF AN ARBITRARY NODE

Consider a node at $(0, 0)$. We will call any other node a neighbor of this node if there is a direct connection between the two nodes. First we obtain the probability distribution of $N_{c,n}(r_0)$, the number of neighbors of the node at $(0, 0)$ that lie in $B(0, 0; r_0)$ for some large $r_0$. If $N_{c,n}'$ is the number of neighbors of the node at $(0, 0)$ over the entire plane, then its mgf is given by
\[
\phi_{N_{c,n}}(s) = \lim_{r_0 \to \infty} \phi_{N_{c,n}'(r_0)}(s).
\]

A. Nodes are points of a homogeneous Poisson process

Applying Theorem 1 to the case where the objects are nodes, the region is $B(0, 0; r_0)$, and the desired property is that of having a connection to $(0, 0)$, we see that $N_{c,n}'(r_0)$, the number of nodes in $B(0, 0; r_0)$ that have a connection to $(0, 0)$, has the following distribution:
\[
N_{c,n}'(r_0) \sim \text{Pois}(\lambda_c \pi r_0^2 p_h(r_0)),
\]
where $p_h(r_0)$, the probability that an arbitrary node whose location is uniformly distributed over $B(0, 0; r_0)$ has a connection to $(0, 0)$, is given by
\[
p_h(r_0) = \mathbb{P} \left\{ R < \min[r_0, r_0X^{2/\delta}e^{hZ/\delta}] \right\} = \mathbb{E}_X, Z \left[ \min \left\{ 1, \left( \frac{r_0}{r_0} \right)^2 X^{4/\delta}e^{hZ/\delta} \right\} \right].
\]
From \((24)\), we then obtain
\[
\phi_{N'_n}(s) = \exp \left\{ -\lambda_n \pi [r_n \exp(\alpha^2)]^2 \mathbb{E}[X^{4/\delta}](1-s) \right\} = q_n^{1-s},
\]
(25)
or \(N'_n \sim \text{Poiss}(\lambda_n \pi [r_n \exp(\alpha^2)]^2 \mathbb{E}[X^{4/\delta}])\).

**B. Nodes are points of a clumped Poisson process**

Recall that the number of clump centers in \(B(0,0; r_0)\) is \(N'_c(r_0) \sim \text{Poiss}(\lambda_c \pi r_0^2)\). Given \(N'_c(r_0) = n_c\), let \(K_1, \ldots, K_{n_c}\) denote the number of nodes in these \(n_c\) clumps that have a direct connection to \((0,0)\). Thus the number of nodes in \(B(0,0; r_0)\) having a direct connection to \((0,0)\) is given by
\[
N'_n(r_0) = \sum_{i=1}^{N'_c(r_0)} K_i.
\]

From our modeling assumptions, it follows that \(\{K_i\}_{i=1}^{\infty}\) are i.i.d. and independent of \(N'_c(r_0)\). Then the same steps as in the proof of Theorem 1 show that the mgf of \(N'_n(r_0)\) is given by
\[
\phi_{N'_n(r_0)}(s) = \phi_{N'_c(r_0)}(\phi_{K_1}(s)) = \exp \left\{ -\lambda_c \pi r_0^2 [1 - \phi_{K_1}(s)] \right\}.
\]
(26)

Now, \(K_1\) is the number of nodes belonging to an arbitrary clump [whose center is in \(B(0,0; r_0)\)] which have a direct connection to \((0,0)\). Recall that the number of nodes in a clump is Poiss(\(\mu_c\)). From Theorem 1 it follows that conditioned on the clump center being at a distance of \(r = r_0\) from \((0,0)\), the distribution of \(K_1\) is Poiss(\(\mu_c p_{c,a}(r)\)), where \(p_{c,a}(r)\) is given by (20). The conditional mgf of \(K_1\) is therefore \(\exp[-\mu_c p_{c,a}(r)(1-s)]\). Integrating over the pdf of \(R\), we may write the unconditional mgf of \(K_1\) as follows:
\[
\phi_{K_1}(s) = \mathbb{E} \left[ e^{-\mu_c (1-s)p_{c,a}(R)} \right] = \int_{0}^{r_0} \frac{2r}{r_0^2} e^{-\mu_c (1-s)p_{c,a}(r)} dr.
\]
(27)

Substituting (27) into (26), and taking the limit as \(r_0 \to \infty\), we have
\[
\phi_{N'_n}(s) = \lim_{r_0 \to \infty} \exp\left\{-\lambda_c \pi r_0^2 [1 - \phi_{K_1}(s)]\right\} = \lim_{r_0 \to \infty} \exp\left\{-\lambda_c \pi r_0^2 \mathbb{E} \left[ 1 - e^{-\mu_c (1-s)p_{c,a}(R)} \right] \right\} = \exp\left\{-2\lambda_c \pi \int_{0}^{\infty} r \left[ 1 - e^{-\mu_c (1-s)p_{c,a}(r)} \right] dr \right\}.
\]
(28)

**VIII. PROBABILITY OF t-HOP OUTAGE**

**A. The probability that a node cannot connect to a BS in \(\leq 2\) hops**

Given \(k\) nodes that have a direct connection to the node at \((0,0)\), we study the probability that an arbitrary BS has no connection to any of these nodes [including the node at \((0,0)\)].

Recall that \(N''_n\) is the number of other nodes in the entire plane to which the node at \((0,0)\) has a connection. Suppose \(N''_n = k\), and the polar coordinate locations of these nodes are \((R_1, \Theta_1), \ldots, (R_k, \Theta_k)\). In the sequel, when we talk of \(k + 1\) nodes, we mean these \(k\) nodes and the node at \((0,0)\).

Let the region be \(B(0,0; r_0)\), the objects be the BSs in \(B(0,0; r_0)\), and the desired property be that a BS have a connection to at least one of the \(k + 1\) nodes, conditioned on these \(k\) nodes in turn having a connection to the node at \((0,0)\). Since this property is independent across BSs and occurs with the same probability \(p_k(r_0)\) for each BS, we see from (5) that the probability that there is no BS in the plane with this property is given by
\[
u_k = \exp \left[ -\lambda_{\text{BS}} \pi \lim_{r_0 \to \infty} r_0^2 p_k(r_0) \right].
\]
(29)

Note that \(p_k(r_0)\), the probability that a BS whose location is uniformly distributed over \(B(0,0; r_0)\) has a direct connection to at least one of the \(k + 1\) nodes, i.e., lies in the union of coverage disks centered at the locations of these \(k + 1\) nodes, is given by
\[
\begin{align*}
p_k(r_0) &= \int_{0}^{2\pi} d\theta_1 \int_{0}^{\infty} dr_1 f_{R_1, \Theta_1} (|R_1| < r_1 X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta))(r_1, \theta_1) \times \cdots \times \int_{0}^{2\pi} d\theta_k \int_{0}^{\infty} dr_k f_{R_k, \Theta_k} (|R_k| < r_k X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta))(r_k, \theta_k) \times \\
&\quad \frac{1}{\text{area}[B(0,0; r_0)]} \mathbb{E} \left\{ \text{area} \left[ B \left( 0, 0; r_{\text{BS}} X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta) \right) \right] \right\} \left\{ \bigcup_{i=1}^{k} B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{\text{BS}} X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta) \right) \right\}.
\end{align*}
\]
(30)

where \((X_1, Z_1), \ldots, (X_k, Z_k)\) are respectively the fast and shadow fade attenuations from the \(k\) nodes to the node at \((0,0)\), and \((X_{\text{BS},0}, Z_{\text{BS},0}), (X_{\text{BS},1}, Z_{\text{BS},1}), \ldots, (X_{\text{BS},k}, Z_{\text{BS},k})\) are respectively the fast and shadow fade attenuations from the node at \((0,0)\) and the \(k\) other nodes to the BS.

Now, the disks \(B(r_i \cos \theta_i, r_i \sin \theta_i; r_{\text{BS}} X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta))\) are not disjoint in general. However, there is no analytical means of determining the area of the union of an arbitrary number of arbitrarily placed disks if the number of these disks is greater than 3. Thus (30) cannot be computed exactly, and we must therefore resort to approximations. We will show two approximations that lead to upper bounds on the area of the union of the disks.

**B. Approximating the area of the union of disks in (30)**

(A1) Approximating the area of the union of the disks by the sum of their areas: Here, we assume that the \(k\) disks are disjoint, and thereby get an upper bound on the area of the union of the disks. From (30),
\[
\pi r_0^2 p_k(r_0) \leq \mathbb{E} \left\{ \text{area} \left[ B \left( 0, 0; r_{\text{BS}} X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta) \right) \right] \right\} + \sum_{i=1}^{k} \int_{0}^{2\pi} d\theta_i \int_{0}^{\infty} dr_i \times f_{R_i, \Theta_i} (|R_i| < r_i X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta))(r_i, \theta_i) \times \mathbb{E} \left\{ \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{\text{BS}} X_{2/\delta}^2 \exp(h Z_{1/\delta}/\delta) \right) \right] \right\}.
\]
which when substituted into (29) yields

\[ u_k \geq \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \mathbb{E} \left[ X_{BS,0}^{A/\delta} e^{2h_{ZBS,0}/\delta} \right] \right\} \]

\[ \times \prod_{i=1}^{k} \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \int_0^{2\pi} d\theta_i \int_0^{\infty} dr_i \times f_{R_i,\theta_i,R_i<r_{BS}X_{BS,i}^{A/\delta}}(r_i,\theta_i) \times \mathbb{E} \left[ X_{BS,1}^{A/\delta} e^{2h_{ZBS,1}/\delta} \right] \right\} \]

\[ = q_{BS} \left( q_{BS}^{(0)} \right)^k, \quad (31) \]

where

\[ q_{BS}^{(0)} = \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \int_0^{2\pi} d\theta \int_0^{\infty} dr \times f_{R,\theta,R<r_{BS}X_{BS}^{A/\delta}}(r,\theta) \times \mathbb{E} \left[ X_{BS,1}^{A/\delta} e^{2h_{ZBS,1}/\delta} \right] \right\} \]

\[ = q_{BS}. \]

In other words, from (31) we see that the assumption of disjoint disks is equivalent to the assumption that the events that the BS has no connection to each of the \( k \) nodes are independent [conditioned on these \( k \) nodes having a connection to the node at \( (0,0) \)], and have the same probability of occurrence, which is given by \( q_{BS}^{(0)} \equiv q_{BS} \). Then from (31), we have \( u_k \geq q_{BS}^{k+1} \).

(A2) Approximating the area of the union of the disks by assuming that the disks are disjoint except with the disk centered at \( (0,0) \): Here, we assume that all disks centered at the \( k \) nodes that have a connection to the node at \( (0,0) \) are mutually disjoint, but have nonempty intersections with the disk centered at \( (0,0) \). In other words, in (30), we upper-bound the area of the union of the disks as follows:

\[ \text{area} \left[ B \left( 0,0; r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] \]

\[ \bigcup_{i=1}^{k} \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{BS}X_{BS,i}^{2/\delta} e^{h_{ZBS,i}/\delta} \right) \right] \]

\[ \leq \text{area} \left[ B \left( 0,0; r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] \]

\[ + \sum_{i=1}^{k} \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{BS}X_{BS,i}^{2/\delta} e^{h_{ZBS,i}/\delta} \right) \right] \]

\[ - \sum_{i=1}^{k} \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{BS}X_{BS,i}^{2/\delta} e^{h_{ZBS,i}/\delta} \right) \right] \]

\[ \cap B \left( 0,0; r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right). \quad (32) \]

Then (30) yields

\[ \pi r_0^2 \rho_k(r_0) \leq \mathbb{E} \left\{ \text{area} \left[ B \left( 0,0; r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] \right\} \]

\[ + \sum_{i=1}^{k} \int_0^{2\pi} d\theta_i \int_0^{\infty} dr_i f_{R_i,\theta_i,R_i<r_{BS}X_{BS,i}^{2/\delta}}(r_i,\theta_i) \times \mathbb{E} \left[ X_{BS,1}^{2/\delta} e^{2h_{ZBS,1}/\delta} \right] \]

\[ \times \left\{ \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{BS}X_{BS,i}^{2/\delta} e^{h_{ZBS,i}/\delta} \right) \right] \right\} \]

\[ - \text{area} \left[ B \left( r_i \cos \theta_i, r_i \sin \theta_i; r_{BS}X_{BS,i}^{2/\delta} e^{h_{ZBS,i}/\delta} \right) \right] \]

\[ \cap B \left( 0,0; r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right). \quad (33) \]

This in turn when substituted into (29) yields

\[ u_k \geq \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \mathbb{E} \left[ X_{BS,0}^{A/\delta} e^{2h_{ZBS,0}/\delta} \right] \right\} \]

\[ \times \left( \prod_{i=1}^{k} \exp \left\{ -\lambda_{BS} \pi r_{BS}^2 \int_0^{2\pi} d\theta_i \int_0^{\infty} dr_i \times f_{R_i,\theta_i,R_i<r_{BS}X_{BS,i}^{2/\delta}}(r_i,\theta_i) \times \mathbb{E} \left[ X_{BS,1}^{2/\delta} e^{2h_{ZBS,1}/\delta} \right] \right\} \right) \]

\[ \times \exp \left\{ -\lambda_{BS} \int_0^{2\pi} d\theta \int_0^{\infty} dr \times f_{R,\theta,R<r_{BS}X_{BS}^{2/\delta}}(r,\theta) \times \mathbb{E} \left[ g \left( r; r_{BS}X_{BS,1}^{2/\delta} e^{h_{ZBS,1}/\delta}, r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] \right\} \]

\[ = q_{BS} \left( q_{BS}^{(1)} \right)^k, \quad (34) \]

where

\[ q_{BS}^{(1)} = q_{BS}^{0} \beta_{BS} = q_{BS} \beta_{BS}, \]

and

\[ \beta_{BS} = \exp \left\{ \lambda_{BS} \int_0^{2\pi} d\theta \int_0^{\infty} dr \times f_{R,\theta,R<r_{BS}X_{BS}^{2/\delta}}(r,\theta) \right\} \]

\[ \times \mathbb{E} \left[ g \left( r; r_{BS}X_{BS,1}^{2/\delta} e^{h_{ZBS,1}/\delta}, r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] \]

\[ = \exp \left\{ \lambda_{BS} \int_0^{\infty} f_{R|R<r_{BS}X_{BS}^{2/\delta}}(r) \times \mathbb{E} \left[ g \left( r; r_{BS}X_{BS,1}^{2/\delta} e^{h_{ZBS,1}/\delta}, r_{BS}X_{BS,0}^{2/\delta} e^{h_{ZBS,0}/\delta} \right) \right] dr \right\}. \quad (35) \]

In other words, the assumption of the \( k \) disks being disjoint except for having a nonempty intersection with the disk centered at \( (0,0) \) leads to the result, seen from (34), that the events that the BS has no connection to each of the \( k \) nodes are independent [conditioned on these \( k \) nodes having a connection to the node at \( (0,0) \), and on the node at \( (0,0) \) having no connection to the BS], and have the same probability of occurrence, which is given by \( q_{BS}^{(1)} \equiv q_{BS} \beta_{BS} \). The difference between \( q_{BS}^{(1)} \) and \( q_{BS}^{(0)} \equiv q_{BS} \) is that the former is the probability under
the additional condition that the node at \((0, 0)\) has no connection to the BS.

From (35), we see that

\[
1 < \beta_{BS} \\
\leq \exp \left\{ \lambda_{BS} \int_0^\infty dr_1 f_{R_1|R_1<r_1 \exp(h_{Z_1} / \delta)}(r_1) \right\} \\
\times \text{area} \left[ B \left( 0, 0; r_{BS} \exp h_{Z_{BS,0}} / \delta \right) \right] \\
= \exp \left\{ \lambda_{BS} \pi \left[ r_{BS} \exp (\alpha^2) \right]^2 E \left[ X_{BS,0}^4 \right] \right\} \\
= 1 / q_{BS},
\]

so we have

\[
q_{BS}^{(0)} < q_{BS}^{(1)} < 1.
\]

Thus (34) is a valid lower bound on \(u_k\), and is tighter than (31).

C. Expressions for the probability that a node cannot connect to a BS in \(\leq 2\) hops

If we have

\[
u_k \geq q_{BS} \beta_{BS} k,
\]

where \(q_{BS}\) is either \(q_{BS}^{(0)}\) or \(q_{BS}^{(1)}\), then we can write the probability of the event that the node at \((0, 0)\) has no connection to any BS within 2 hops as follows:

\[
\sum_{k=0}^\infty \mathbb{P} \{ N_n' = k \} u_k \geq q_{BS} \sum_{k=0}^\infty \left( \frac{q_{BS}}{q_{BS}'} \right)^k \\
= q_{BS} \phi_{N_n'}(q_{BS}) = q_2,
\]

so (36)

\[
\mathbb{P} \{ N_n' = k \} \geq \mathbb{P} \{ N_n' = k \} q_{BS}^{(1)},
\]

In other words, \(q_2\) as defined by (36) is the lower bound on the probability that an arbitrary node cannot connect to a BS in \(\leq 2\) hops. We now compute \(q_2\) for the two choices of \(q_{BS}\) corresponding to \(q_{BS}^{(0)}\) and \(q_{BS}^{(1)}\) derived from the above approximations A1 and A2 respectively.

(a) For the approximation A1 above, where we have \(q_{BS} = q_{BS}^{(0)}\), we obtain:

(i) Nodes are points of a homogeneous Poisson process: Here

\[
N_n' \sim \text{Poiss} \left( \lambda_n [r_n \exp(\alpha^2)]^2 E[X_{BS,0}^4] \right),
\]

so from (36), we have

\[
q_2 = q_{BS} \exp \left\{ -\lambda_n [r_n \exp(\alpha^2)]^2 E[X_{BS,0}^4] \right\} \\
\times (1 - q_{BS}) \\
= q_{BS}^2 h^{1 - q_{BS}}.
\]

(ii) Nodes are points of a clumped Poisson process: Here we have

\[
\phi_{N_n'}(s) = \exp \left\{ -2\lambda_n \pi \int_0^\infty r \\
\times (1 - \exp[-\mu_c(1 - s)p_{c,n}(r)]) \, dr \right\},
\]

and (36) yields

\[
q_2 = q_{BS} \exp \left\{ -2\lambda_n \pi \int_0^\infty r \\
\times (1 - \exp[-\mu_c(1 - q_{BS}p_{c,n}(r)]) \, dr \right\},
\]

where \(p_{c,n}(r)\) is given by (19).

(b) For the approximation A2, which yields \(q_{BS} = q_{BS}^{(1)}\), we obtain in the same way as above:

(i) Nodes are points of a homogeneous Poisson process:

\[
q_2 = q_{BS} \exp \left\{ -\lambda_n [r_n \exp(\alpha^2)]^2 E[X_{BS,0}^4] \right\} \\
\times (1 - q_{BS}) \\
= q_{BS}^2 h^{1 - q_{BS}^{(1)}}.
\]

(ii) Nodes are points of a clumped Poisson process:

\[
q_2 = q_{BS} \exp \left\{ -2\lambda_n \pi \int_0^\infty r \\
\times (1 - \exp[-\mu_c(1 - q_{BS}^{(1)}(q_{BS}^{(1)})^2)]) \, dr \right\}.
\]

D. Expressions for the probability that a node cannot connect to a BS in \(\leq t\) hops, where \(t > 2\)

In this section, we generalize the analysis of the previous section, where we focused on the special case of \(t = 2\). First, we note that

\[
\mathbb{P} \{ \text{None of } k \text{ arbitrary nodes has a direct connection to any BS} \mid \text{These } k \text{ nodes have a direct connection to the node at } (0, 0), \text{ which also has no direct connection to any BS} \}
\]

\[
= \mathbb{P} \{ \text{Neither any of } k \text{ arbitrary nodes, nor the node at } (0, 0), \text{ has a direct connection to any BS} \mid \text{These } k \text{ nodes have a direct connection to the node at } (0, 0) \}
\]

\[
= \frac{u_k}{q_{BS}}, \quad (39)
\]

Then, given our assumption (in approximations A1 and A2 above) that conditioned on the node at \((0, 0)\) having no direct connection to any BS, the events that these \(k\) nodes have direct connections to these BSs are mutually independent, we recognize from (39) that

\[
\mathbb{P} \{ \text{An arbitrary node has no direct connection to any BS} \}
\]

\[
= \hat{\beta}_{BS} \mathbb{P} \{ \text{An arbitrary node has no direct connection to any BS} \}, \quad (40)
\]

where

\[
\hat{\beta}_{BS} = \begin{cases} 
\beta_{BS}^{(0)}, & \text{under A1}, \\
\beta_{BS}^{(1)}, & \text{under A2}.
\end{cases}
\]

We may now derive the \(t\)-hop outage probability in the same way as in the left hand side of (36):

\[
q_t = \mathbb{P} \{ \text{Node at } (0, 0) \text{ cannot connect to any BS} \}
\]

\[
\text{in } \leq t \text{ hops}.
\]
\[ q_{BS} \sum_{k=0}^{\infty} \mathbb{P}\{N'_n = k\} \left( \mathbb{P}\{\text{An arbitrary node has no direct connection to any BS in } \leq t - 1 \text{ hops } | \text{ It has a direct connection to the node at } (0,0), \text{ and the node at } (0,0) \text{ has no direct connection to any BS}\} \right)^k \]

\[ = \sum_{k=0}^{\infty} \mathbb{P}\{N'_n = k\} \left( \mathbb{P}\{\text{An arbitrary node has no direct connection to any BS } | \text{ It has a direct connection to the node at } (0,0), \text{ and the node at } (0,0) \text{ has no direct connection to any BS}\} \right)^k \times \mathbb{P}\{\text{This arbitrary node has no direct connection to any node that can connect to a BS in } \leq t - 2 \text{ hops } | \text{ It has a direct connection to the node at } (0,0), \text{ and neither it nor the node at } (0,0) \text{ has a direct connection to any BS}\} \left( \mathbb{P}\{\text{An arbitrary node has no direct connection to any BS } | \text{ It has a direct connection to the node at } (0,0), \text{ and the node at } (0,0) \text{ has no direct connection to any BS}\} \right)^k \]

\[ = q_{BS} \sum_{k=0}^{\infty} \mathbb{P}\{N'_n = k\} \left( \mathbb{P}\{\text{An arbitrary node has no direct connection to any BS } | \text{ It has a direct connection to the node at } (0,0), \text{ and the node at } (0,0) \text{ has no direct connection to any BS}\} \right)^k \times \left( \hat{\beta}_{BS} \mathbb{P}\{\text{An arbitrary node has no direct connection to any BS}\} \right)^k \]

\[ = q_{BS} \phi_{N'_n}(q_{t-1}). \]  

As before, results on \( q_{t} \) can now be derived for both homogeneous and clumped Poisson processes. Note that for approximation A2 and the homogeneous Poisson process, the form of the mgf for \( N'_n \) shows that

\[ q_t = q_{BS} q_n^{1 - \hat{\beta}_{BS} q_{t-1}}, \quad t \geq 2, \]  

which is the generalization of (38).

**IX. Expressions for \( f_{R|R<r_0, X^{2/\delta}}(r) \)**

In order to use the expressions derived for \( q_2 \) and \( q_t \) in the previous sections, it remains only to compute \( \hat{\beta}_{BS} \), which from (35) requires a knowledge of the pdf of the distance \( R \) of an arbitrary node from the node at \((0,0)\), conditioned on there being a connection between the two nodes, i.e., on \( R < r_0 X^{2/\delta} \exp(\delta r/h) \).

**A. Nodes are points of a homogeneous Poisson process**

Consider a node at \((0,0)\), and the disk \( B(0,0; r_0) \), where \( r_0 \) is assumed very large. Consider an arbitrary node in \( B'(0,0; r_0) \), and let \( A_{r_0} \) denote the event that this node has a direct connection to the node at \((0,0)\). We are interested in the distribution of the distance \( R \) of this node from \((0,0)\), given that it has a direct connection to the node at \((0,0)\). In other words, we wish to evaluate the conditional cdf

\[ F_{R|A_{r_0}}(r) = \mathbb{P}\{R \leq r | A_{r_0}\} = \frac{\{R \leq r\} \cap A_{r_0}}{\mathbb{P}(A_{r_0})}, \]  

(44)

Clearly, the denominator of (44) is

\[ \mathbb{P}(A_{r_0}) = \mathbb{P}\{R < r_0 X^{2/\delta} \exp(\delta r/h)\}. \]

Since the nodes are points of a homogeneous Poisson process, it follows that the location of any node that is known to be in \( B'(0,0; r_0) \) must be uniformly distributed over \( B'(0,0; r_0) \). In other words, the (unconditional) pdf of \( R \) is given by

\[ f_R(r) = \begin{cases} \frac{2r}{r_0^2}, & 0 \leq r \leq r_0, \\ 0, & r > r_0. \end{cases} \]

thereby yielding

\[ \mathbb{P}(A_{r_0}) = \mathbb{P}\{R < r_0 X^{2/\delta} \exp(\delta r/h)\} \]

\[ \to \frac{1}{r_0} \mathbb{E}[X^{2/\delta}] \text{ as } r_0 \to \infty, \]  

(45)

where we follow the same steps as in the derivation of (13). The numerator of (44) may be evaluated as follows:

\[ \mathbb{P}\{ \{R \leq r\} \cap A_{r_0}\} = \mathbb{P}\{R \leq r, R < r_0 X^{2/\delta} \exp(\delta r/h)\} \]

\[ = \int_{0}^{\min(r_0, r)} 2r' \int_{-\infty}^{\infty} \frac{\exp[-z^2/(2r'^2)]}{\sigma \sqrt{2\pi}} \frac{1}{2} \mathbb{E}[X^{2/\delta}] \int_{-\infty}^{\infty} \int_{0}^{\min(r_0, r)} f_X(x) \frac{dxdzdr'}{r_0^2} \]

\[ \times \left\{ 1 - \mathbb{E}[F_X(\sqrt{r_0^2 + h^2})] \right\} dr'. \]  

(46)

Now, let \( A \) denote the event that an arbitrary node (anywhere in the plane) has a direct connection to a node at \((0,0)\). Then the desired conditional cdf \( F_{R|A}(r) \) is obtained by substituting (46) and (45) into (44) and taking the limit as \( r_0 \to \infty \), and differentiating yields the conditional pdf

\[ f_{R|A}(r) = \frac{2r \left\{ 1 - \mathbb{E}[F_X(\sqrt{r_0^2 + h^2})] \right\}}{[r_0 \exp(\alpha^2)]^2 \mathbb{E}[X^{2/\delta}]]. \]  

(47)
B. Nodes are points of a clumped Poisson process

We begin by determining the pdf of the distance \( R \) of an arbitrary node from the origin. Let the center of the clump to which this node belongs be at a distance of \( R \) from the origin. We know the pdf of \( R \) to be given by

\[
f_R(\tilde{r}) = \frac{2\tilde{r}}{r_0^2}, \quad 0 \leq \tilde{r} \leq r_0.
\]

Suppose we want to find \( F_R(r) \). If \( r \leq r_c \), the radius of a clump, then the node is at distance \( r \) or less from the origin if and only if its clump center lies at a distance of at most \( r + r_c \) from the origin, and the node itself lies in the part of the clump that intersects with the disk of radius \( r \) centered at the origin. In other words,

\[
F_R(r) = \int_0^{r + r_c} f_R(\tilde{r}) \frac{g(\tilde{r}; r, r_c)}{\pi r_c^2} d\tilde{r}, \quad r \leq r_c.
\]  (48)

Differentiating with respect to \( r \), we obtain

\[
f_R(r) = f_R(r + r_c) \frac{g(r + r_c; r, r_c)}{\pi r_c^2} + \int_0^{r + r_c} f_R(\tilde{r}) \frac{1}{\pi r_c^2} \frac{\partial g(\tilde{r}; r, r_c)}{\partial r} d\tilde{r}, \quad r \leq r_c.
\]  (49)

Similarly, if \( r > r_c \) and \( r_0 \) sufficiently large, then the node is at a distance \( r \) or less from the origin if and only if either the clump center is at a distance of \( r - r_c \) or less from the origin, or the clump center is at a distance of between \( r - r_c \) and \( r + r_c \) from the origin, and the node itself lies in the part of the clump that intersects with the disk of radius \( r \) centered at the origin. The cdf is therefore

\[
F_R(r) = f_R(r - r_c) + \int_{r - r_c}^{r + r_c} f_R(\tilde{r}) \frac{g(\tilde{r}; r, r_c)}{\pi r_c^2} d\tilde{r}, \quad r > r_c,
\]  (50)

and the corresponding pdf is

\[
f_R(r) = f_R(r - r_c) \left[ 1 - \frac{g(r - r_c; r, r_c)}{\pi r_c^2} \right] + f_R(r + r_c) \frac{g(r + r_c; r, r_c)}{\pi r_c^2} + \int_{r - r_c}^{r + r_c} f_R(\tilde{r}) \frac{1}{\pi r_c^2} \frac{\partial g(\tilde{r}; r, r_c)}{\partial r} d\tilde{r}, \quad r > r_c.
\]  (51)

Defining the function \( x_+ = \max\{x, 0\} \), we may combine (48)-(51) to write

\[
F_R(r) = \frac{1}{r_0^2} \left( \left( r - r_c \right)^2 + \int_{r - r_c}^{r + r_c} 2r \frac{g(\tilde{r}; r, r_c)}{\pi r_c^2} d\tilde{r} \right),
\]

\[
f_R(r) = \frac{1}{r_0^2} \left( 2r - r_c \right) + \left[ 1 - \frac{g(r - r_c; r, r_c)}{\pi r_c^2} \right] + \int_{r - r_c}^{r + r_c} \frac{1}{\pi r_c^2} \frac{\partial g(\tilde{r}; r, r_c)}{\partial r} d\tilde{r},
\]  (52)

for all \( r \geq 0 \) if \( r_0 \) is sufficiently large. We may then write

\[
\mathbb{P}(A_{r_0}) = \mathbb{P}\left\{ R < r_n X^{2/\delta} e^{hZ/\delta} \right\}
\]

and follow the derivation of (46) to obtain

\[
\mathbb{P}(\{ R \leq r \} \cap A_{r_0}) = \int_0^{\min(r, r_c)} f_R(r') \times \left\{ 1 - \mathbb{E}_Z \left[ F_X \left( \sqrt{e^{-hZ} (r'/r_n)\delta} \right) \right] \right\} dr',
\]

thereby yielding the conditional pdf to be

\[
f_{R|A}(r) = \left( \mathbb{E}_X, Z \left\{ \left( (r' - r_c)^2 + \int_{(r'-r_c)}^{(r'+r_c)} 2\tilde{r} \frac{g(\tilde{r}; r, r_c)}{\pi r_c^2} d\tilde{r} \right)^{-1} \right. \right)
\]

\[
\times \left( \mathbb{E}_X, Z \left\{ \left[ 2(r - r_c) + \int_{(r-r_c)}^{(r+r_c)} \frac{1}{\pi r_c^2} \frac{\partial g(\tilde{r}; r, r_c)}{\partial r} d\tilde{r} \right] \right. \right)
\]

\[
\times \left\{ 1 - \mathbb{E}_Z \left[ F_X \left( \sqrt{e^{-hZ} (r/r_n)\delta} \right) \right] \right\},
\]  (53)

where \( x_+ = \max\{x, 0\} \).

X. Numerical Results

We ran simulations to compare the analytic bounds with simulation results. We set \( \delta = 4 \) and \( r_\text{BS} = r_n = 1000 \) m. In other words, when \( \sigma = 0 \) dB the limiting reception range is 1000 m on all links. We also fix \( \lambda_\text{BS} = 10^{-7} \text{m}^{-2} \).

A. Homogeneous Poisson Model

For the homogeneous Poisson node location model, for each choice of \( \lambda_n \), the simulation makes “partial” [14] realizations of the homogeneous Poisson process by repeatedly placing \( 100\lambda_n a^2 \) nodes independently and uniformly over a large square of size \( 10a \) for some chosen \( a \). Similarly, we place \( 100\lambda_\text{BS} a^2 \) BSs over the large square. There are i.i.d. fades (shadow and Rayleigh) between all pairs of nodes. To eliminate the dependence on edge effects, we collected statistics only on the nodes in a small disk (with radius \( a \)) at the center of the large square.

In Fig. 1, we plot \( q_2, q_3, \) and \( q_4 \) from simulations with \( \sigma = 8 \) dB and Rayleigh fading. As an illustration of the dependence of the outage probability on the wireless channel parameters, we have also plotted \( q_2 \) for \( \sigma = 0 \) dB with Rayleigh fading. The difference in the plots for \( \sigma = 0 \) dB and \( \sigma = 8 \) dB clearly demonstrates that the fading must be taken into account in such analyses. Unfortunately, computing \( \beta_\text{BS} \) for this choice of parameters from (35) and (47) requires a 6-dimensional numerical integration, which could not be accomplished in a reasonable time by the chosen integration package (Matlab). In Fig. 2, we have therefore plotted \( q_2 \) for the above setup (with
\( \lambda \) obtained from simulations vs. \( \sigma = 8 \) dB, a homogeneous Poisson process. The BS density is \( \lambda_{BS} = 10^{-2} \text{m}^{-2} \), and other parameters are \( r_{BS} = r_n = 1 \text{km} \), and \( \delta = 4 \).

\( \sigma = 8 \) dB but with no fast fading [i.e., \( X = X_{BS,0} = X_{BS,1} \equiv 1 \) in (47) and (35)] and compared it to the two analytic lower bounds (37) and (38) obtained from approximations A1 and A2 respectively. As expected, A2 yields a significantly tighter lower bound than A1.

### B. Clumped Poisson Model

For the case where the node locations are points of a clumped Poisson process, a partial realization is generated as follows: first generate \( n_c = 100 \lambda_c a^2 \) clump center locations instead of node locations following the above scheme. Then populate the region with \( \mu_c n_c \) nodes whose locations are chosen as follows: for each node, choose one of the \( n_c \) clump centers at random, then draw the node location coordinates from the uniform distribution in a disk of radius \( r_c \) centered at the chosen clump center. In Fig. 3, we plot \( q_2 \), \( q_3 \), and \( q_4 \) obtained from simulations vs. \( \lambda_c \) for this setup, with Rayleigh fading, shadow fading with \( \sigma = 8 \) dB, \( r_c = 500 \text{ m} \), and \( \mu_c = 5 \). Unfortunately, even without Rayleigh fading, the evaluation of \( \beta_{BS} \) from (53) and (35) requires 5-dimensional numerical integration, which also could not be done in a reasonable time, so a comparison of the simulation and analytical lower bounds could not be provided at time of submission.

### XI. Conclusions

Allowing nodes to serve as repeaters for data packets generated by or destined to nodes that do not have a direct connection to any BS extends the reach of BSs in cellular architectures, thereby improving coverage. Since similar benefits could also be achieved by other means, e.g., deploying more BSs, using higher towers at the BSs, or boosting the transmit power, it is important to evaluate what can be achieved with multihop designs. In this paper we investigate scenarios where the BSs are distributed randomly and uniformly over the service area, the nodes are either also distributed randomly and uniformly, or according to the clumped Poisson point process, and the propagation channels exhibit signal attenuation with distance and Rayleigh, Ricean, or Nakagami fading. We calculate the probability that a potential node will not have access to any BS, show how it depends on the density of BSs and nodes, and how it diminishes as more hops are allowed. We have developed exact analytical expressions and concise lower bounds when this was not possible. In the later case, we have produced simulation results for comparison. We demonstrate that the use of nodes as repeaters is particularly beneficial in fading environments.

### ACKNOWLEDGMENTS

We wish to thank an anonymous reviewer for drawing our attention to the fact that (6) follows rigorously from the
monotone convergence of probability measures from above as $r_0 \rightarrow \infty$. We would also like to thank Constantinos Papadias and Paul Polakos for their support of this research.

REFERENCES


