# Node-Disjoint Paths in Hierarchical Hypercube Networks

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## Abstract

*The hierarchical hypercube network is suitable for massively parallel systems. An appealing property of this network is the low number of connections per processor, which can facilitate the VLSI design and fabrication of the system. Other alluring features include symmetry and logarithmic diameter, which imply easy and fast algorithms for communication. In this paper, a maximal number of node-disjoint paths are constructed between every two distinct nodes of the hierarchical hypercube network. Their maximal length is not greater than* $\max\{2^{m+1}+2m+1, 2^{m+1}+m+4\}$*, where* $2^{m+1}$ *is the diameter.*

## 1. Introduction

Advances in hardware technology, especially the VLSI circuit technology, have made it possible to build a large-scale multiprocessor system that contains thousands or even tens of thousands of processors. For example, the Connection Machine [6] contains as many as $2^{16}$ processors. One crucial step on designing such a multiprocessor system is to determine the topology of the interconnection network (network for short), because the system performance is significantly affected by the network topology. In recent decades, a number of networks have been proposed in the literature [1, 4, 5, 7, 8, 10, 12, 13]. A network is conveniently represented by a graph (or digraph) whose vertices represent the nodes (i.e., processors) of the network and whose edges represent the communication links of the network. Throughout this paper, we use network and graph, node and vertex, and edge and link interchangeably.

Let $G = (V, E)$ be a connected graph, where $V$ and $E$ represent the vertex set and edge set of $G$, respectively. The *degree* of a vertex in $G$ is the number of edges incident with it. If all vertices have the same degree $d$, then $G$ is called *regular* or $d$*-regular*. The *distance* between two vertices $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path between $u$ and $v$. The *diameter* of $G$ is the maximal distance between any two vertices. The *vertex* (*edge*) *connectivity* of $G$ is the minimal number of vertices (edges) in $G$ whose removal can cause $G$ disconnected or trivial.

An $n$-dimensional hypercube, denoted by $Q_n$, is one of the most popular networks. There are $2^n$ nodes contained in $Q_n$; each is uniquely represented by a binary sequence $b_{n-1}b_{n-2}...b_0$ of length $n$. Two nodes in $Q_n$ are adjacent if and only if they differ at exactly one bit position. An edge of $Q_n$ is said *of dimension* $k$ ($0 \le k \le n-1$) if its two end vertices differ at $b_k$. The hypercube suffers from a practical limitation when it is used as the topology of a multiprocessor system. As $n$ increases, it becomes more difficult to design and fabricate the nodes of $Q_n$ because of the large fanout.

To remove the limitation, the cube-connected cycles (CCC) network [10] was designed as a substitute for the hypercube. The node degree of CCC is restricted to three. However, this restriction degrades the performance of CCC at the same time. For example, CCC has a larger diameter than the hypercube. Taking both the practical limitation and the performance into account, the hierarchical hypercube (HHC) network [9] was proposed as a compromise between the hypercube and CCC. HHC, which has a two-level structure, takes hypercubes as basic modules and connects them in a hypercube manner. HHC has a logarithmic diameter, which is the same as the hypercube. Since the topology of HHC is closely related to the topology of the hypercube, HHC inherits some favorable properties from the

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hypercube.

Suppose that $A$ and $B$ are two distinct nodes of a graph $G$. An $(A, B)$-*container* in $G$ is a set of internally node-disjoint paths (disjoint paths for short) between $A$ and $B$. According to Menger's theorem [2], there are $\kappa$ disjoint paths between $A$ and $B$, where $\kappa$ is the node connectivity of $G$. The *width* (*length*) of a container is the number (maximal length) of paths it contains. In this paper, a container with maximal width (i.e., $\kappa$) is constructed between two arbitrary nodes of HHC. The container has length not greater than $\max\{2^{m+1}+2m+1, 2^{m+1}+m+4\}$, where $2^{m+1}$ is the diameter. In the next section, the structure of HHC is first reviewed. Then the container is constructed in Section 3. Finally, this paper concludes with some remarks in Section 4.

## 2. HHC Networks

Recall that CCC can be obtained by replacing each node of $Q_k$ with a cycle of $k$ nodes so that these $k$ nodes are connected to the $k$ neighbors of the original node in $Q_k$. Actually, HHC is a modification of CCC in which the $k$-node cycle is replaced with a hypercube. Assume $k = 2^m$. HHC can be constructed as follows: start with $Q_{2^m}$ and replace each node of $Q_{2^m}$ with $Q_m$. Refer to Figure 1, where an example with $m = 2$ is shown. Since there are $2^{2^m} \times 2^m = 2^{2^m+m}$ nodes in HHC, each node of HHC can be uniquely represented by a binary sequence $b_{n-1}b_{n-2}...b_0$, where $n = 2^m + m$. For convenience, $b_{n-1}b_{n-2}...b_0$ is expressed as a two-tuple $(S, P)$, where $S = b_{n-1}b_{n-2}...b_m$ tells which $Q_m$ the node is located in and $P = b_{m-1}b_{m-2}...b_0$ gives the address of the node in the located $Q_m$.

Let $P^{(l)} = b_{m-1}...b_{l+1}\bar{b}_l b_{l-1}...b_0$ ($S^{m+l} = b_{n-1}...b_{m+l+1}\bar{b}_{m+l}b_{m+l-1}...b_m$) denote the binary sequence obtained by complementing $b_l$ ($b_{m+l}$) of $P$ ($S$). HHC can be defined in terms of graph as follows.

**Definition 2.1** *The node set of HHC is* $\{(S, P)|$ *for all* $S = b_{n-1}b_{n-2}...b_m$ *and* $P = b_{m-1}b_{m-2}...b_0\}$, *where* $n = 2^m + m$ *and* $m \geq 1$. *Node adjacency of HHC is defined as follows:* $(S, P)$ *is adjacent to* (1) $(S, P^{(l)})$ *for all* $0 \leq l \leq m-1$ *and* (2) $(S^{(m+dec(P))}, P)$, *where* $dec(P)$ *is the decimal value of* $P$.

Edges defined by (1) are referred to as *internal edges*, and those defined by (2) are referred to as *external edges*. Internal edges are within $Q_m$ and each of external edges connects two $Q_m$'s. Since each node of HHC has the same degree $m + 1$, HHC is $(m + 1)$-regular. Moreover, HHC is symmetric and has a diameter of $2^{m+1}$ (see [9]). In subsequent discussion, whenever a node $A$ of HHC is mentioned, we use $A_S$ and $A_P$ to represent the $S$ part and $P$ part of $A$, respectively.

## 3. Containers

Suppose that $A$ and $B$ are two distinct nodes of HHC. In this section, a maximal number (i.e., $m + 1$) of disjoint paths from $A$ to $B$ are constructed. Since HHC is symmetric, we assume $A = (0^{2^m}, 0^m)$ without losing generality, where $0^{2^m}$ ($0^m$) represents $2^m$ ($m$) consecutive 0's. These paths contain internal edges and external edges alternately. Those internal edges between two external edges are within the same $Q_m$. Since each path is desired to be as short as possible, each subpath within $Q_m$ is maintained shortest. It is easy to obtain a shortest path between any two nodes of $Q_m$. So, if the subpaths within $Q_m$'s are ignored, then a path in HHC can be simply represented by a sequence of external edges, called an *external edge sequence* (EES).

For example, $A = (0000, 00) \xrightarrow{*} (0000, 11) \rightarrow (1000, 11) \xrightarrow{*} (1000, 10) \rightarrow (1100, 10) \xrightarrow{*} (1100, 01) \rightarrow (1110, 01) = B$ expresses a path from $A = (0000, 00)$ to $B = (1110, 01)$, where $\xrightarrow{*}$ denotes a shortest path within $Q_2$. The path contains three external edges which can be denoted by their $P$ parts, i.e., 11, 10 and 01 in sequence. Hence the path can be simply represented by an EES (11, 10, 01). Any path from $A$ to $B$ contains at least $|B_S|$ external edges, where $|B_S|$ is the number of bits 1 in $B_S$. An EES is *shortest* if it contains $|B_S|$ external edges. In order to reduce the path length, a particular shortest EES, denoted by $\pi$, is obtained from $B_S$, as described below.

First, collect the indices $i$ of $B_S = b_{2^m+m-1}b_{2^m+m-2}...b_m$ with $b_i = 1$. Second, decrease each $i$ by $m$, and so $0 \leq i \leq 2^m - 1$. Third, construct $\pi$ by arranging all indices $i$ (expressed in binary form) into a subsequence of an $m$-bit Gray code [3]. The latter consists of $2^m$ codewords in which every two adjacent codewords differ at exactly one bit position. For example, when $m = 3$ and $B_S = 10101111$, the set of indices obtained after the second step is $\{0, 1, 2, 3, 5, 7\}$. Since a 3-bit Gray code can be $(000, 001, 011, 010, 110, 111, 101, 100)$, we have $\pi = (000, 001, 011, 010, 111, 101)$ finally. We assume in the rest of this section that an $m$-bit Gray code begins with $0^m$.

Suppose that $t$ is an $m$-bit binary sequence contained in $\pi$. Define $\pi^t$ to be the shortest EES that is obtained by cyclically shifting $\pi$ toward the left until the resulting EES begins with $t$. For example, if $\pi = (000, 001, 011, 010, 111, 101)$, then $\pi^{001} = (001, 011, 010, 111, 101,$
$000)$ and $\pi^{111} = (111, 101, 000, 001, 011, 010)$. There are paths that contain more than $|B_S|$ external edges. EESs that represent such paths are not shortest. For

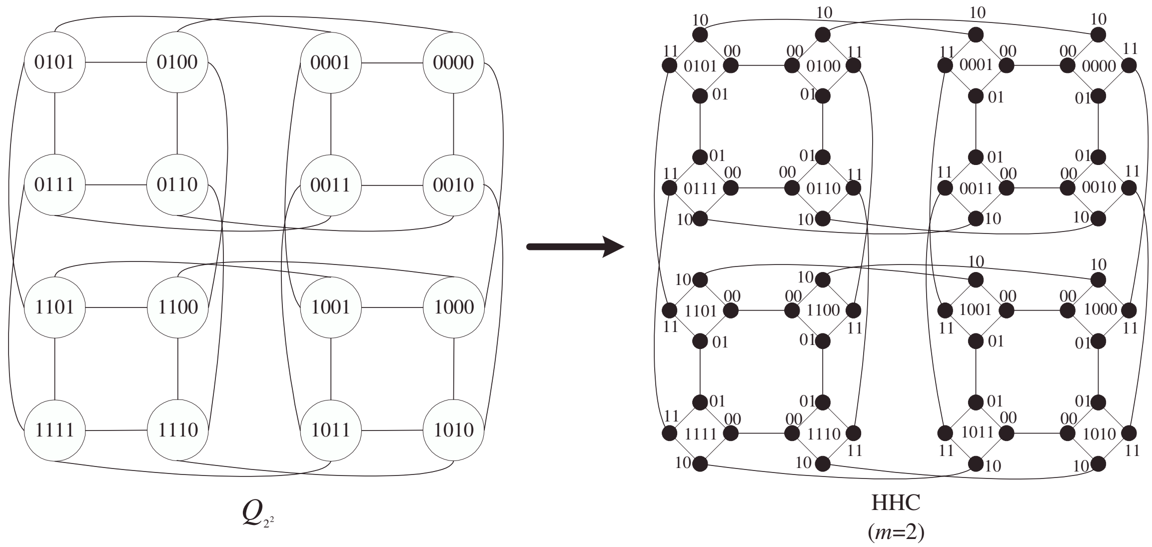


**Figure 1. Construction of HHC from $Q_{2^2}$.**

example, $A = (0000, 00) \xrightarrow{*} (0000, 11) \rightarrow (1000, 11) \xrightarrow{*} (1000, 00) \rightarrow (1001, 00) \xrightarrow{*} (1001, 11) \rightarrow (0001, 11) = B$ contains $3 > 1 = |B_S|$ external edges, and so its corresponding EES, i.e., (11, 00, 11), is not shortest. It will be clear later that EESs can greatly help the construction of disjoint paths.

Now we begin to describe the construction method. First, we consider a special situation in which $A$ and $B$ are located within the same $Q_m$, i.e., $A_S = B_S$. According to Saad and Schultz's construction method [12], $m$ disjoint paths from $A$ to $B$ can be obtained within the $Q_m$. Moreover, these $m$ disjoint paths have maximal length not greater than $m + 1$. One more disjoint path is constructed according to the EES $(0^m, B_P, 0^m, B_P)$. In the rest of this section, we assume $A_S \neq B_S$.

Since HHC is $(m + 1)$-regular, each edge incident with $A$ or $B$ is included in some disjoint path. Suppose $\pi = (c_0, c_1, ..., c_{r-1})$. One or two disjoint paths, which contain the two external edges that are incident with $A$ or $B$, are constructed below, depending on four cases.

Case 1. $A_P \in \pi$ and $B_P \in \pi$. We have $c_0 = 0^m$. Assume $c_z = B_P$, where $0 \leq z \leq r - 1$. Two (or one if $z = r - 1$) disjoint paths are constructed according to $\pi^{c_0}$ and $\pi^{c_{(z+1) \bmod r}}$.

Case 2. $A_P \notin \pi$ and $B_P \in \pi$. Suppose that $\tau = (0^m, c_0, c_1, ..., c_{r-1})$ is a subsequence of an $m$-bit Gray code. Two disjoint paths are constructed according to $(\tau, 0^m)$ and $\pi^{c_{(z+1) \bmod r}}$, where $c_z = B_P$ is assumed.

Case 3. $A_P \in \pi$ and $B_P \notin \pi$. We have $c_0 = 0^m$. Suppose that $\theta = (c_0, ..., c_{u-1}, B_P, c_u, ..., c_{r-1})$ is a subsequence of an $m$-bit Gray code, where $0 \leq u \leq r$. One disjoint path is constructed according to $(\theta^{B_P}, B_P)$, where $\theta^{B_P}$ is obtained by cyclically shifting $\theta$ toward the left until the resulting EES begins with $B_P$. Another disjoint path is constructed according to $\pi^{c_0}$.

Case 4. $A_P \notin \pi$ and $B_P \notin \pi$. One disjoint path is constructed according to $(\tau, 0^m)$, where $\tau$ was defined in Case 2. If $B_P \neq 0^m$, another disjoint path is constructed according to $(\theta^{B_P}, B_P)$, where $\theta^{B_P}$ was defined in Case 3.

Next, the other $m-1$ disjoint paths are constructed, as described below. For simplicity, $A_P \notin \pi$ and $B_P \in \pi$ (i.e., Case 2) are assumed; the discussion for the other cases is similar. If $r \geq m$, they are constructed according to the first $m-1$ unused EESs of $\pi^{c_0}, \pi^{c_1}, ..., \pi^{c_{r-1}}$. If $r < m$, then $r - 1$ disjoint paths are constructed according to the unused EESs of $\pi^{c_0}, \pi^{c_1}, ..., \pi^{c_{r-1}}$. The remaining $m - r$ disjoint paths are constructed by the aid of Rabin's work [11], as explained below.

Arbitrarily select $m - r$ neighboring nodes of $A$ within the same $Q_m$ so that they were not included in $\pi$. Without losing generality, assume that $A_P^{(0)}$, $A_P^{(1)}$, ..., $A_P^{(m-r-1)}$ are selected. For $0 \leq l \leq m - r - 1$, suppose that $\rho_l = (c_0, ..., c_{w_l - 1}, A_P^{(l)}, c_{w_l}, ..., c_{r-1})$ is a subsequence of an $m$-bit Gray code, where $0 \leq w_l \leq r$. The remaining $m - r$ disjoint paths are constructed according to $(\rho_l^{A_P^{(l)}}, A_P^{(l)})$'s, where $\rho_l^{A_P^{(l)}}$ is the EES obtained by cyclically shifting $\rho_l$ toward the left until the resulting EES begins with $A_P^{(l)}$. It was shown in [11] that there are $m$ disjoint paths from $A$ ($B$) to $A_P^{(0)}, A_P^{(1)}, ..., A_P^{(m-r-1)}$, $c_0, c_1, ..., c_{r-1}$, respectively, whose maximal length is not greater than $m+1$.

There are $m+1$ paths constructed above. They are disjoint because they traversed distinct $Q_m$'s, exclusive of the first $Q_m$ and the last $Q_m$ where $A$ and $B$ reside, respectively. Their lengths are computed as follows. The paths obtained by $\pi^{c_0}, \pi^{c_1}, ..., \pi^{c_{r-1}}$ have lengths not greater than $r+(m+1)+(2^m-2)+(m+1)$, where $r$ is the number of external edges traversed and $m+1$ is an upper bound on the numbers of internal edges traversed in the first and last $Q_m$'s. Since $A_P = 0^m$ is not contained in $\pi$, the total number of internal edges traversed in the other $Q_m$'s is not greater than $2^m-2$.

The path obtained by $(\tau, 0^m)$ has length not greater than $(r+2)+2^m+(m+1)$, as explained below. There are $r+2$ external edges traversed. No internal edges were traversed in the first $Q_m$ and not more than $m+1$ internal edges were traversed in the last $Q_m$. At most $2^m$ internal edges were traversed in the other $Q_m$'s. Similarly, the path obtained by $(\rho_l^{A_P^{(l)}}, A_P^{(l)})$ has length not greater than $(r+2)+1+2^m+(m+1)$. At most $2^m$ internal edges were traversed in all $Q_m$'s but the first $Q_m$ and the last $Q_m$. To sum up, the $m+1$ paths above have maximal length equal to $\max\{2^m+2m+r, 2^m+m+r+4\}$.

The discussion (path construction and length computation) above is based on Case 2. For the other cases, the discussion is similar. The constructed $m+1$ disjoint paths have maximal length equal to $\max\{2^m+2m+r+1, 2^m+m+r+4\}$, $\max\{2^m+2m+r+1, 2^m+m+r+4\}$ and $\max\{2^m+2m+r, 2^m+m+r+4\}$, if Case 1, Case 3 and Case 4 are considered, respectively. Consequently, the $m+1$ disjoint paths from $A$ to $B$ have maximal length equal to $\max\{2^m+2m+r+1, 2^m+m+r+4\} \leq \max\{2^{m+1}+2m+1, 2^{m+1}+m+4\}$. Therefore, we have the following theorem.

**Theorem 3.1** *There exists a container of width $m+1$ between any two distinct nodes of an HHC with $2^{2^m+m}$ nodes whose length is not greater than $\max\{2^{m+1}+2m+1, 2^{m+1}+m+4\}$, where $2^{m+1}$ is the diameter.*

## 4. Conclusion

It is practically important to construct containers because they can be used to increase the transmission rate and to enhance the transmission reliability. In this paper, we have constructed a container of width $m+1$ in an hierarchical hypercube network with $2^{2^m+m}$ nodes. The length of the container, which is greater than the diameter ($= 2^{m+1}$) by $\max\{2m+1, m+4\}$, counts the number of internal edges and external edges traversed.

The number of external edges traversed is $r$ or $r+2$ ($r = |B_S|$), where the number of internal edges traversed depends on the EESs used. In our analysis of container length, we made a worst-case estimation on both numbers. We computed $r = 2^m$ and the number of internal edges traversed with respect to an EES to be $2^m-1$. Actually, $d_H(c_i, c_{i+1})$ internal edges are traversed for any two adjacent $m$-bit binary sequences $c_i$ $c_{i+1}$ in an EES, where $d_H(c_i, c_{i+1})$ denotes the Hamming distance between $c_i$ and $c_{i+1}$. It seems that there is a tradeoff between the value of $r$ and the number of internal edges with respect to an EES. For example, when $r$ approaches $2^m$ very few internal edges are traversed. It is an interesting problem how to reach an optimization with this tradeoff.

In our analysis of container length, we consider $r = 2^m$ and the number of internal edges required with respect to an EES is $2^m-1$, both worst-case estimation.

Since the size of $B_S$ and the number of the first (last) $Q_m$ are trade off. When $|B_S|$ is large enough, it means we have more choices of EES. We use the upper bound to estimate the size of $|B_S|$. Moreover, we use the upper bound in the first (last) $Q_m$ again. For this reason, the upper bound of the container is over estimation. The upper bound of the container can be improved. A tighter bound is possible and we leave it to the future work.

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