Smoothed Form of Nonlinear Phase Macromodel for Oscillators

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***Abstract* — This paper proposes an improvement to the well-known oscillator nonlinear phase macromodel based on Floquet theory. A smoothed form of the nonlinear phase macromodel is derived by eliminating highly oscillatory terms in the macromodel, resulting in a significant speed-up in transient simulation. For an LC oscillator under sinusoidal excitation the new macromodel is equivalent to the Adler model. Numerical experiments confirm a considerable decrease of computational efforts. It is further shown that the new macromodel allows one to perform phase noise analysis of locked oscillators under arbitrary periodic injection.**

I. INTRODUCTION

In many practical applications the analysis of electronic circuits with oscillators can be simplified and accelerated if the system of oscillator equations is replaced by a phase macromodel. At the present time two main approaches are used to construct a phase macromodel of the oscillator: the Adler equation [1] and the nonlinear phase equation based on Floquet theory [2].

In the classical Adler paper [1] an LC oscillator under sinusoidal excitation is considered, and a differential equation for the slowly varying phase of the oscillator waveform is derived.

The Adler equation can be applied to LC oscillators only. Following Adler's approach, similar equations were obtained for some other oscillator circuits [3], injection-locked frequency dividers [4, 5], coupled oscillators [6], and ring oscillators [7]. Each of these cases started with analytical macromodels for the particular oscillator. Hence, this approach cannot be implemented into a circuit simulator to provide the analysis of an arbitrary circuit using accurate transistor models.

In contrast, the nonlinear phase macromodel based on Floquet theory [2] can be applied to an arbitrary oscillator circuit. In many practical applications the replacement of the full oscillator model by the first order differential equation of the macromodel provides transient simulation without notable loss of the accuracy. The high efficiency of this approach was demonstrated in the analysis of injection locking behavior of oscillators [8, 9], simulation of coupled oscillators [10], and analysis of Phase Locked Loops (PLL) [11, 12].

However the macromodel has a drawback in comparison with the Adler model. The phase variable of the macromodel obtained by transient simulation contains small high frequency variations along a smooth curve [13]. These variations add nothing to the information on the circuit behavior but require a large number of redundant simulation steps. Moreover, such variations prevent the determination of the DC solution of the macromodel differential equation, which in turn prevents its application to standard small-signal, stability, and noise analyses near the DC operating point.

In this paper we propose a modification to the nonlinear phase macromodel [2] that removes parasitic variations from the solution and provides smooth phase and frequency waveforms. The obtained smoothed macromodel is suitable for phase noise analysis of locked oscillator.

The rest of the paper is organized as follows. In Section II the differential equations corresponding to the Adler model and to the nonlinear phase macromodel of a weakly nonlinear LC oscillator under sinusoidal excitation are investigated. It is shown that in comparison with the Adler model the nonlinear phase macromodel contains an additional highly oscillatory term. This term is the cause of parasitic high frequency variations observed in phase and frequency waveforms computed using the macromodel.

In Section III the general case of any oscillator circuit under arbitrary slow varying periodic excitation is considered. It is shown that in this case the nonlinear phase macromodel can also be presented in the form containing oscillatory terms. A smoothed form of the nonlinear phase macromodel is derived by neglecting the oscillatory terms. It is demonstrated that the smoothed macromodel reduces to known models in special cases. In particular the Adler equation is obtained from the smoothed macromodel for the oscillator under sinusoidal excitation.

The application of the smoothed macromodel to perform stability and phase noise analysis of locked oscillators is presented in Section IV.

Section V is devoted to the mathematical analysis of the error arising due to neglecting highly oscillatory terms in the nonlinear phase macromodel. It is shown that the smoothed macromodel provides the same accuracy order as the original nonlinear phase macromodel for the cases considered in the paper. In the general case the accuracy order is retained if the variations of input waveforms are sufficiently small in adjacent periods of free running oscillations.

In Section VI some experimental results are presented. Transient simulations demonstrate more than the tenfold reduction of computational efforts in comparison with the nonlinear phase macromodel. The results of phase noise analysis of locked oscillator are also presented in the section.

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II. MOTIVATION

The nonlinear phase differential equation based on Floquet theory is presented in the form [2]

$$\frac{d\alpha}{dt} = v^T(t+\alpha)b(t) \tag{1}$$

where *v(t)* is a perturbation projection vector (PPV), *b(t)* is an external excitation, α is the unknown phase variable.

The components of the PPV are periodic functions, which can be represented as Fourier series

$$v_l(t) = \sum_{k=-K}^{K} V_{kl} e^{jk\omega_0 t} \tag{2}$$

Here v_l is a component of the PPV corresponding to the node *l*. Harmonics V_{kl} satisfy conjugate condition $V_{-kl} = V_{kl}^*$.

To perform the comparison with the Adler model we consider the case of an LC oscillator excited by a sine wave applied to the one input node

$$b_{inp}(t) = A\sin((\omega_0 + \Delta\omega)t) \tag{3}$$

Here A, $(\omega_0 + \Delta\omega)$ are the magnitude and frequency of excitation respectively, and ω_0 is the fundamental of free-running oscillator.

It was shown [14] that for an LC oscillator the components of the PPV are sinusoidal functions, and therefore the component of the input node is $v_{inp}(t) = 2|V_{1,\,inp}|\cos(\omega_0(t+\alpha))$. Hence (1) for the LC oscillator can be written as follows:

$$\frac{d\alpha}{dt} = 2A|V_{1,\,inp}|\cos(\omega_0(t+\alpha))\sin((\omega_0+\Delta\omega)t) \tag{4}$$

Here the coefficient 2 appears due to the usage of double-sided Fourier series in (2). After $\alpha(t)$ is determined from (4) the oscillator waveform without amplitude deviations is evaluated in accordance with [2] by

$$x(t) = x^0(t+\alpha(t)) \tag{5}$$

where $x^0(t)$ is the waveform of free running oscillator.

The waveform of the LC oscillator is $x^0(t) = A_{osc}\sin(\omega_0 t)$, so (5) is transformed to

$$x(t) = A_{osc}\sin(\omega_0 t + \phi(t)), \tag{6}$$

where

$$\phi(t) = \omega_0\alpha(t), \tag{7}$$

The Adler equation for an LC oscillator under small sinusoidal injection with frequency $\omega_{inj} = \omega_0 + \Delta\omega$ has the form [1]

$$\frac{d\theta}{dt} = -\Delta\omega - \frac{\omega_0}{2Q}\cdot\frac{A}{A_{osc}}\sin(\theta) \tag{8}$$

Here $\theta(t)$ is slow-varying phase of oscillator waveform, Q is the quality factor of the LC tank, A, A_{osc} are the magnitudes of the injection signal and capacitance voltage, respectively. In accordance with the analysis presented in [14] it can be shown that the first PPV harmonic of an LC oscillator at the input node has the magnitude

$$|V_{1,\,inp}| = \frac{A}{2QA_{osc}} \tag{9}$$

and the phase is equal to π. Thus the expression (8) can be written in the form

$$\frac{d\theta}{dt} = -\Delta\omega + \omega_0 A|V_{1,\,inp}|\sin(\theta) \tag{10}$$

In the Adler model the waveform of the LC oscillator under excitation is assumed to be defined as

$$x(t) = A_{osc}\sin(\omega_{inj}t - \theta(t)) \tag{11}$$

After comparison of (11) and (6) one can see that

$$\phi(t) = \Delta\omega\cdot t - \theta(t) \tag{12}$$

To compare the phase behavior of the two models the numerical integration of (4), (10) and subsequent computation of the phase variable $\phi(t)$ by (7), (12) were performed. The magnitude of the sinusoidal injection was sufficient to provide locking conditions of LC oscillator at $\Delta\omega = 0.05\omega_0$. Simulation results are presented in Fig. 1. One can see that $\phi(t)$ obtained by the Adler model (10, 12) is a linear function with the slope corresponding to a frequency deviation. The nonlinear phase macromodel (4, 7) generates small oscillations along the line.

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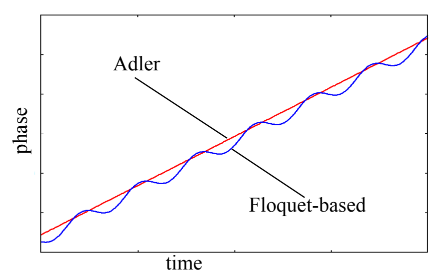

Fig. 1. Phase waveforms evaluated by various models.

More clearly the difference between models is seen from Fig. 2 where the time dependence of the instantaneous frequency is shown

$$\omega_{inst} = \omega_0 + d\phi/dt \tag{13}$$

The curve obtained by (4, 7) contains unrealistic oscillations with frequency about $2\omega_0$. Note that the definition of the instantaneous frequency (13) is well-grounded only for slow-varying $\phi(t)$. Thus, one can expect that highly oscillatory deviations in the $d\phi/dt$ (Fig. 2) do not represent frequency behavior but define small variations in the waveform (5). This is confirmed by the comparison of the oscillator output waveforms (6) and (11) shown in Fig. 3. The maximum difference is about 4%. It is much less than the difference between the magnitudes of free-running and locked oscillator representing the amplitude error of the macromodel (1, 5) in comparison with the full oscillator model.

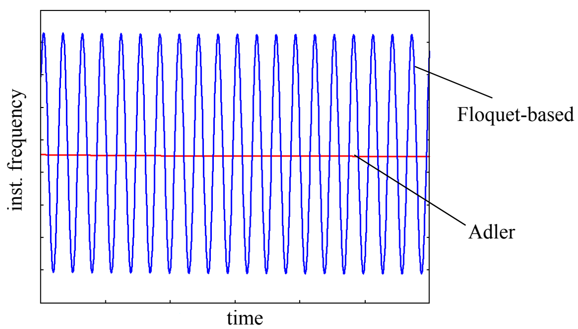

Fig. 2. The instantaneous frequency waveforms evaluated by various models.

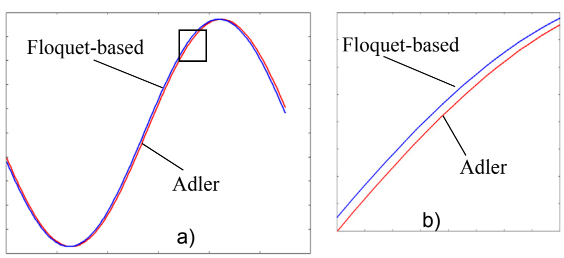

Fig. 3. Output waveforms evaluated by various models (a), fragment (b)

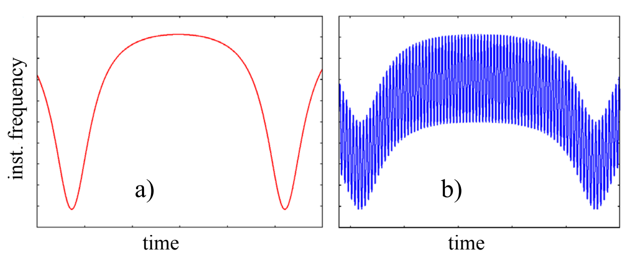

Fig. 4. The instantaneous frequency in the injection pulling mode evaluated by Adler (a) and nonlinear phase (b) models.

The obtained experimental results allow us to conclude that both macromodels provide similar accuracy. Moreover, the Adler model has the following advantages:

1. Numerical solution of (4) requires far fewer integration steps than the solution of (10). In the LC example the ratio of the number of steps was 7420/ 608 =12.3.

2. The Adler model is much more suitable to the analysis of the instantaneous frequency of the oscillator under excitation. For example the analysis of injection pulling by both models is presented in Fig. 4.

3. Injection locking conditions can be easily obtained from (8) by setting $d\theta/dt = 0$ [1]. Subsequent small-signal and phase noise analysis of locked oscillator can also be easily performed [5].

However, unlike the nonlinear phase macromodel the Adler model cannot be applied to an arbitrary circuit. Thus, it is desirable to perform the transformation of (1) to eliminate the parasitic oscillations in the solution for any oscillator circuit. To achieve this aim we need to determine the source of the oscillations.

Taking into account that the solution of the Adler equation does not contain the parasitic oscillations one can conclude that the source of the oscillations can be found by the comparison of (4) and (10). To perform the comparison the equations must be presented with respect to common variable $\phi(t)$. Substituting (7) into (4) and (12) into (10) we obtain

- for the nonlinear phase macromodel

$$\frac{d\phi}{dt} = \omega_0 A|V_{1,\,inp}|\sin(\Delta\omega t - \phi) + \omega_0 A|V_{1,\,inp}|\sin((2\omega_0 + \Delta\omega)t + \phi) \tag{14}$$

- for the Adler model

$$\frac{d\phi}{dt} = \omega_0 A|V_{1,\,inp}|\sin(\Delta\omega \cdot t - \phi) \tag{15}$$

The expression (14) in contrast with (15) contains the additional highly oscillatory term. From the above numerical experiments one can conclude that this term has little effect on the accuracy of the model, but essentially impairs the numerical integration and the representation of results. Thus the Adler model (15) can be considered as a smoothed form of the nonlinear phase macromodel for an LC oscillator (14).

Our aim is to determine a smoothed form for the general nonlinear phase macromodel (1) which will be applicable to any oscillator.

III. Smoothed Macromodel Equation

A. *Derivation of the Equation*

In this section we consider the oscillator behavior under excitation defined by harmonics of the oscillator fundamental with time-dependent slowly varying phasors $B_{ml}(t)$ and a slowly varying DC vector $B_0(t)$. Then the component of the excitation at lth node is represented as slowly varying double-sided Fourier series

$$b_l(t) = \frac{1}{2}\sum_{m \neq 0} B_{ml}(t)\exp j(m\omega_0 t) + B_0(t) \tag{16}$$

Taking into account (2), (16) we can evaluate the dot product in the right hand side of (1) as follows

$$v^T(t+\alpha)\cdot b(t) = \sum_{k,l} V_{kl}B_{0l}(t)\exp(jk\omega_0 t) + \frac{1}{2}\sum_{k,l,m} V_{kl}B_{ml}(t)\exp(j((k+m)\omega_0 t + m\omega_0\alpha)) \quad (17)$$

Note that oscillatory terms are represented in (17) by the exponents with $k+m \neq 0$. It was shown in Section II that such term in LC oscillator introduces only small amplitude deviation in the solution of (1) and can be neglected in phase analysis. Removing these terms we obtain the equality

$$v^T(t+\alpha)\cdot b(t) = V_0^T B_0(t) + Re\Big(\sum_l \sum_{k>0} V_{kl}B_{kl}(t)\exp(-jk\omega_0\alpha)\Big) \quad (18)$$

Replacing the phase variable by (7) and substituting (18) into (1) we obtain

$$\frac{1}{\omega_0}\frac{d\phi}{dt} = W(B(t), \phi(t)) + V_0^T \cdot B_0(t) \quad (19)$$

where

$$W(B(t), \phi) = Re\Big(\sum_l \sum_{k>0} V_{kl}\cdot B_{kl}(t)\exp(-jk\phi)\Big) \quad (20)$$

or in the real form

$$W(B(t), \phi) = \sum_l \sum_{k>0} |B_{kl}(t)||V_{kl}|\cos(\psi_{kl} + \varphi_{kl}(t) - k\phi) \quad (21)$$

where $\varphi_{kl}(t)$, ψ_{kl}, are angles of phasors $B_{kl}(t)$, V_{kl}.

In some cases it is more convenient to represent the excitation as the sum of excitations (16) with magnitudes $B^{(n)}$ (*n=1,2,...*). Then the phase model is described by the equation

$$\frac{1}{\omega_0}\frac{d}{dt}(\phi) = \sum_n (W(B^{(n)}(t), \phi(t)) + V_0^T \cdot B_0^{(n)}(t)) \quad (22)$$

B. *Some Special Cases*

Having obtained the general equation (19) we can derive useful expressions for some special cases of oscillators under excitation and compare them with well-known results.

When the oscillator is excited by an unmodulated sine wave with frequency near *m*th harmonic ($\omega_{inj} = m\omega_0 + \Delta\omega$) the excitation (16) can be considered as the signal with the frequency $m\omega_0$ and slow linearly varying phase ($\varphi_{m,inp} = \Delta\omega \cdot t$). Then for the excitation with magnitude A applied to the one node, the expressions (19, 21) are transformed to

$$\frac{d\phi}{dt} = \omega_0 A|V_{m,inp}|\cos(\psi_{m,inp} + \Delta\omega\cdot t - \phi(t)) \quad (23)$$

In the case *m=1* (23) practically coincides (except for constant phase shift) with the transformed Adler equation (15). The case *m>1* corresponds to a divide by *m* injection locked frequency divider. The Adler model was first derived for weakly nonlinear LC oscillators. It was expanded to some specific cases of other classes of oscillators in [3-7]. Equation (23) proves that the Adler model is valid for an arbitrary oscillator circuit if the coefficient is defined by the corresponding PPV harmonic.

When the oscillator is excited by a slowly varying DC signal *D(t)* applied to the input node, equation (19) becomes:

$$\frac{d\phi}{dt} = \omega_0 V_{0,inp}\cdot D(t) \quad (24)$$

This expression corresponds with the well-known VCO representation as an ideal integrator in PLL macromodels [15].

In the case *D(t)=const* (24) defines the sensitivity of the oscillator fundamental to the value of DC sources

$$\frac{\Delta\omega}{\omega_0} = V_{0,inp}\cdot D \quad (25)$$

Taking into account that $V_{0,inp} = \frac{1}{T}\int_0^T v_{inp}(t)dt$, it can be easily shown that (24) is equivalent to the integral expression for the sensitivity presented in [16].

The smoothed phase equation (19, 20) can be used for oscillator injection locking analysis under arbitrary periodic excitations with common frequency $\omega_0 + \Delta\omega$ applied to any number of the circuit nodes. In this case slowly varying phasors in (16) are presented as $B_{kl}(t) = B_{kl}\cdot\exp(jk\Delta\omega\cdot t)$, and (19) takes the form

$$\frac{d\phi}{dt} = \omega_0 W(B, \phi(t) - \Delta\omega\cdot t) \quad (26)$$

Note that the steady-state solution of (26) is

$$\phi(t) = \Delta\omega\cdot t + \phi_0, \quad (27)$$

where ϕ_0 satisfies the equation

$$W(B, \phi_0) = \Delta\omega/\omega_0 \quad (28)$$

This coincides with the injection locking phase algebraic equation derived in [17].

IV. Application of Smoothed Macromodel to the Stability, Small-Signal and Phase Noise Analysis of Locked Oscillator

It has been mentioned above that the original nonlinear phase macromodel does not allow one to perform stability and noise analysis of locked oscillator due to nonexistence of the DC solution. Here we apply the smoothed macromodel to solve the problem.

The equation (28) for the phase of a locked oscillator has more than one solution because $W(B, \phi_0)$ is a periodic function with period 2π. The existence of two solutions is illustrated in Fig. 5 where the curve $W(B, \phi_0)$ crosses twice the level of relative offset frequency.

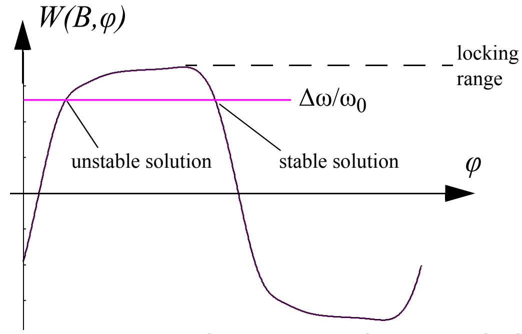

Fig. 5. Two solutions of phase equation for 7-stages CMOS ring oscillator under pulsed excitation

The stability analysis is needed to determine which solution can exist in the circuit. To perform the analysis we must consider small deviations $\eta(t)$ about the DC solution. Thus (27) should be replaced by

$$\phi(t) = \Delta\omega \cdot t + \phi_0 + \eta(t), \tag{29}$$

Substituting (29) into (26) and linearizing the obtained equation with respect to $\eta(t)$, the following equation is obtained

$$\frac{d\eta}{dt} = \omega_0 W'(B, \phi_0)\eta(t) \tag{30}$$

$$\text{where } W' = \frac{dW}{d\phi} = \sum_{k>0}\left(k\sum_{l} V_{kl} \cdot B_{kl}\sin(\varphi_{kl} + \psi_{kl} - k\phi)\right) \tag{31}$$

The stability condition of the first order linear differential equation (30) is evaluated as $W'(B, \phi_0) < 0$. Hence the stable solution in Fig. 5 corresponds to the negative slope of the curve.

To simplify the expressions we perform phase noise analysis for the case when injection frequency is equal to the oscillator fundamental. So we consider two signals exciting the oscillator:

- sufficiently large locking signal with frequency ω_0, represented by the vector of harmonics B,
- small sinusoidal signal applied to nth node with the frequency $m\omega_0 + \delta\omega$, and magnitude B_{mn}^{small}.

Exploiting the expression (22) we obtain

$$\frac{1}{\omega_0}\frac{d\phi}{dt} = W(B, \phi) + Re(V_{mn}B_{mn}^{small}\exp j(\delta\omega \cdot t - m\phi)) \tag{32}$$

Assuming the solution in the form $\phi(t) = \phi_0 + \eta(t)$ (32) yields

$$\frac{1}{\omega_0}\frac{d\eta}{dt} = W(B, \phi_0 + \eta(t)) + Re(V_{mn}B_{mn}^{small}\exp j(\delta\omega \cdot t - m\phi_0 - m\eta(t))) \tag{33}$$

We can linearize rhs of (33) with respect to $\eta(t)$. It can be shown that $\eta(t)$ in the argument of the exponent can be neglected because it corresponds with second order terms in Taylor expansion. Linearizing $W(B, \phi_0 + \eta(t))$ we obtain:

$$\frac{1}{\omega_0}\frac{d\eta}{dt} - W'(B, \phi_0)\eta(t) = Re(V_{mn}B_{mn}^{small}\exp j(\delta\omega \cdot t - m\phi_0)) \tag{34}$$

The equation (34) is a first order differential equation with sinusoidal rhs that provides the transfer factor

$$\left|H_{mn}^{lock}(\delta\omega)\right|^2 = \frac{\omega_0^2|V_{mn}|^2}{(\omega_0 W'(B, \phi_0))^2 + \delta\omega^2} \tag{35}$$

Note that the transfer factor of a free running oscillator corresponds to zero injection magnitudes in (35) and

$$\left|H_{mn}^{free}(\delta\omega)\right|^2 = \frac{\omega_0^2|V_{mn}|^2}{\delta\omega^2} \tag{36}$$

After comparison (35) with (36) one can conclude

$$\left|H_{mn}^{lock}(\delta\omega)\right|^2 = \left|H_{mn}^{free}(\delta\omega)\right|^2\frac{\delta\omega^2}{(\omega_p^2 + \delta\omega^2)}, \tag{37}$$

$$\text{where } \omega_p = \omega_0 W'(\phi_0) \tag{38}$$

The phase noise can be defined by squares of transfer factors multiplied by the power spectral density (PSD) of noise sources. Thus the phase noise in the locked oscillator can be determined by the phase noise in a free running oscillator similar to (37)

$$\left|L_{\phi}^{lock}\right| = \left|L_{\phi}^{free}\right|\frac{\delta\omega^2}{\omega_p^2 + \delta\omega^2} \tag{39}$$

This expression can be found in the literature (see e.g. [5]) but with the definition of ω_p for special cases of oscillator circuits. Our approach provides the evaluation of ω_p for arbitrary oscillators. It can be easily implemented into a circuit simulator.

It must be pointed out that (36) conforms to the linear model of oscillator phase behavior, which may be obtained by neglecting the variable α in the rhs of (1). In contrast to [2], such model (ISF) [18] does not provide the rigorous theory for phase noise. However, as shown in [19] both methods produce equal results for the oscillator's phase noise behavior.

V. Accuracy Evaluation

In this section we present the analysis of error due to neglecting highly oscillatory terms in (17) of the form $V_{kl} \cdot r_{osc}$ where

$$r_{osc} = B_{ml}(t)\exp j(n\omega_0 t + m\omega_0\alpha) \quad (n = k + m \neq 0) \quad (40)$$

Similarly with [2] we assume the excitation norm to be sufficiently small that implies a small value of phase derivative

$$\dot{\alpha}(t) = \frac{d\alpha}{dt} = O(\|b\|) \quad (41)$$

The following estimates can be easily proved

$$B_{ml}(t) = O(\|b\|) \quad (42)$$

$$\alpha(t) - \alpha(0) = O(\|b\|) \text{ at } 0 \le t \le T \quad (43)$$

$$\Delta\alpha = \alpha(T) - \alpha(0) = \int_0^T \dot{\alpha}(t)dt = O(\|b\|) \quad (44)$$

Here $\Delta\alpha$ is the phase variation on the period of free running oscillations (T), and $\dot{\alpha}(t)$ can be replaced by the rhs of (1).

To evaluate the error we can determine the order of the contribution of the term (40) into the phase variation (44)

$$\Delta\alpha_{osc} = \int_0^T r_{osc}(t)dt \quad (45)$$

Representing the phase in the period as

$$\alpha(t) = \alpha(0) + (\alpha(t) - \alpha(0)), \quad (46)$$

and taking into account (43) we can transform (40) by applying Taylor expansion with respect to $\alpha(t) - \alpha(0)$

$$\begin{aligned} r_{osc}(t) &= B_{ml}(t)\exp j(n\omega_0 t + m\omega_0\alpha(0)) + \\ &jm\omega_0(\alpha(t) - \alpha(0))B_{ml}(t)\exp j(n\omega_0 t + m\omega_0\alpha(0)) + O(\|b\|^2) \end{aligned} \quad (47)$$

From (42), (43) one can conclude that the second term in (47) is $O(\|b\|^2)$, and to evaluate (45) it is sufficient to compute the integral of the first term only. This integral can be transformed as

$$\begin{aligned} &jn\omega_0\int_0^T B_{ml}(t)\exp j(n\omega_0 t + m\omega_0\alpha(0))dt = \\ &B_{ml}(T) - B_{ml}(0) + \int_0^T \dot{B}_{ml}(t)\exp j(n\omega_0 t + m\omega_0\alpha(0))dt \end{aligned} \quad (48)$$

The value of (48) is $O(\|b\|^2)$ if

$$\dot{B}_{ml}(t) = O(\|b\|^2) \quad (49)$$

that implies

$$B_{ml}(t + T) - B_{ml}(t) = O(\|b\|^2). \quad (50)$$

Thus if (49) holds then $\Delta\alpha_{osc} = O(\|b\|^2)$, and it is seen that at sufficiently small excitation the error $\Delta\alpha_{osc}$ is much less than the value of the phase deviation (44). Obtained order of $\Delta\alpha_{osc}$ implies the following statement.

The difference between the solutions of initial value problems ($\alpha(0) = \alpha_0$) by the original macromodel (1) and by the smoothed macromodel is evaluated as

$$e(nT) = O(\|b\|^2) \quad (51)$$

where $e(t) = \alpha(t) - \alpha_{sm}(t)$, $\alpha_{sm}(t)$ is the solution of the smoothed equation, n is any integer.

Note that $e(t) = O(\|b\|)$ in the intermediate points $t \neq nT$. But this distinction can be considered as amplitude variations evaluated by the first order Taylor expansion of (5)

$$x^0(t + \alpha_{sm}(t)) = x^0(t + \alpha(t)) - \dot{x}^0(t + \alpha(t))e(t) \quad (52)$$

One can see from (52) that additional amplitude deviations due to the replacement of the original macromodel by the smoothed one are evaluated by the first order value $O(\|b\|)$. It has been proved in [2] that amplitude deviations of the nonlinear phase macromodel with respect to the full oscillator model are also $O(\|b\|)$. Thus the smoothed macromodel provides the same order of the amplitude error.

So oscillatory terms can be neglected if (49) is true. A similar (but more complicated) analysis expands this result to the generalized smoothed equation (22).

To analyze conditions of (49) validity we first consider the excitation with constant magnitude and time-varying phase $B_{ml}(t) = B_{ml}\exp jm\varphi(t)$ when

$$\dot{B}_{ml}(t) = jm\Delta\omega_{inst}B_{ml}\exp jm\varphi(t) \quad (53)$$

where $\Delta\omega_{inst} = \dot{\varphi}(t)$ is the instantaneous frequency offset.

One can see from (53) that (49) is valid under

$$\Delta\omega_{inst} = O(\|b\|) \quad (54)$$

Note that locking conditions obtained from (28) [17] represent the linear dependence of locking range on input harmonics. So the value of locking range satisfies (54), and injection locking analysis by smoothed macromodel provides the same order of accuracy as the analysis by the original macromodel. This result can be expanded to injection pulling analysis when an input frequency is slightly above the locking range and so (54) is also valid.

Expressions (35, 37) for the phase transfer factor of a locked oscillator contain the value of corner offset frequency ω_p that satisfies (54). Thus small-signal and phase noise analysis of a locked oscillator performed in Section IV provide sufficient accuracy in the neighborhood of ω_p, which is the most important range in phase noise analysis of a locked oscillator.

Condition (49) can be considered as a mathematical interpretation of the formulation “slowly varying Fourier series” for the representation of the excitation in the form (16). It is easy to see that if the harmonic envelope is presented in the general exponential form

$$B_{ml}(t) = B_{ml}\exp\beta(t) \qquad (55)$$

with complex $\beta(t)$ then (54) is replaced by

$$\dot{\beta}(t) = O(\|b\|) \qquad (56)$$

It can be shown that (56) is valid for PLL transient analysis [10, 11] if the reference waveform is presented similar to (55).

One can assume that in practice condition (49) or (50) can be interpreted as sufficiently small deviations of the input waveform in adjacent cycles of free running oscillations

$$\|b(t+T) - b(t)\| \ll \|b\| \qquad (57)$$

We can expect that (57) is valid in many applications.

VI. Numerical Experiments

A. *Transient Simulation*

In Section II we presented some results of the experimental comparison of models for weakly nonlinear LC oscillator under sine excitation (taking into account that the Adler model is a special case of the smoothed macromodel).

In this section we present simulation results for several nonlinear oscillators. The following four circuits were taken for experiments: Colpitts oscillator, 3- and 7-stages CMOS ring oscillators, and Bi-CMOS differential ring oscillator from [20].

Table 1 contains relative values of the magnitudes of high PPV harmonics with respect to the magnitude of the first PPV harmonic. It is seen that the PPV for weakly nonlinear LC oscillator (row 1) contains practically only the first harmonic. The relatively larger values of higher harmonics for the other test circuits gives an indication of the degree of nonlinearity (rows 2-5).

Table 1 Characteristics of test circuits

Oscillator	Freq. MHz	Magnitudes of PPV harmonics %			
		2	3	4	5
LC	0.159	0.0001	0.033	0.00007	0.015
Colpitts	3.39	74.1	55.3	37.7	23.4
Ring3	1474	19.2	11.3	3.1	1.0
Ring7	570.4	36.8	63.5	24.3	30.7
Ring_diff	3500	0	50.7	0	27.8

The circuits were tested under pulse train excitation in injection locking mode and in injection pulling mode. The examples of the waveforms for 7-stages CMOS ring oscillator are presented in Fig. 6, 7.

The essential reduction of computational effort due to the application of the smoothed macromodel can be seen in Table 2, where the number of simulation steps is given for both macromodels in injection pulling and injection locking modes.

The simulations were performed by trapezoidal method in the interval [0, 100T], where T is the period of free-running oscillator.

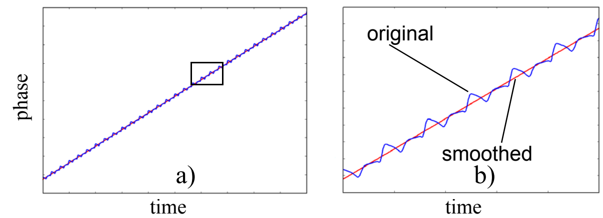

Fig. 6. (a) Phase waveforms in injection locking mode. (b) Fragment

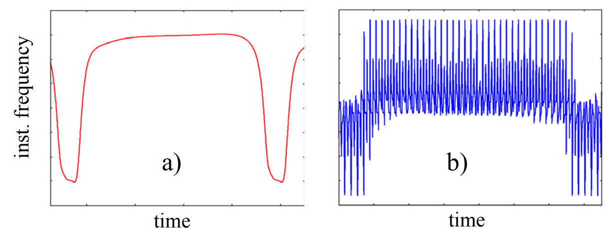

Fig. 7. The instantaneous frequency in the injection pulling mode evaluated by smoothed (a) and original (b) macromodels.

Table 2 Comparison of computational efforts

Test conditions		Number of simulation steps		
Mode	Oscillator	Original	Smoothed	Ratio
Injection pulling	Collpitts	6677	439	15.2
	Ring3	5838	177	38.6
	Ring7	8927	311	28.7
	Ring_diff	7834	409	19.2
Injection locking	Collpitts	6681	298	22.4
	Ring3	8347	199	41.9
	Ring7	9535	248	38.4
	Ring_diff	8732	325	26.7

B. *Phase Noise Example*

The example of phase noise evaluation is presented in Fig. 8 and 9 both for free running oscillator and locked oscillator. The simulation results were obtained for the Bi-CMOS differential ring oscillator. Fig. 8 shows the simulation results of the circuit with white noise sources, and Fig. 9 corresponds to the circuit with flicker noise sources.

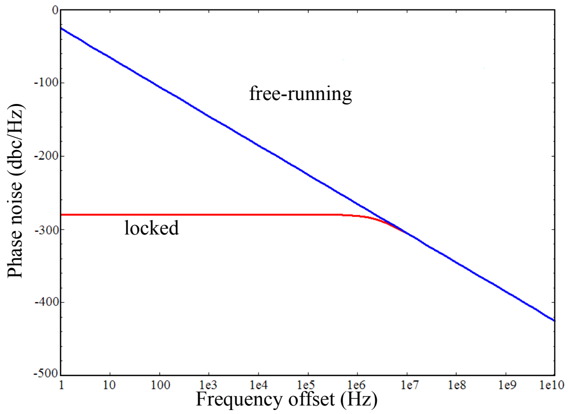

Fig. 8. Phase noise in Bi-CMOS ring oscillator: white noise

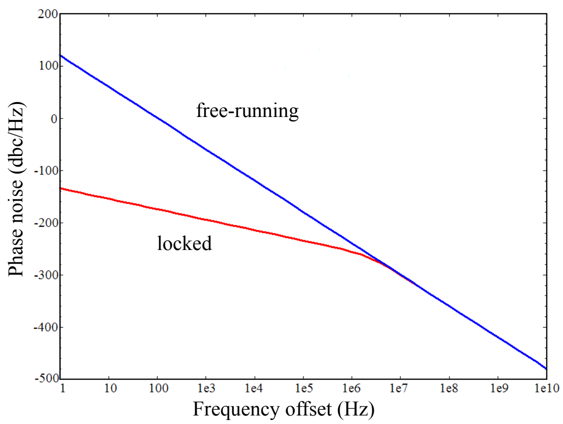

Fig. 9. Phase noise in Bi-CMOS ring oscillator: flicker noise

VII. Conclusion

The smoothed nonlinear phase macromodel provides more than an order of magnitude speedup of transient simulation in comparison with the original macromodel due to eliminating high-order components, without principal lost of accuracy. Unlike the original macromodel the smoothed oscillator macromodel allows one to perform stability analysis and also small-signal and noise analyses of the oscillators locked by arbitrary periodic excitation.

The proposed macromodel can be considered as a natural generalization of the classical Adler model equation and the Adler equation can be obtained as its special case.

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