# On Optimal Quadratic Regulation for Discrete-Time Switched Linear Systems*

Wei Zhang and Jianghai Hu

School of Electrical and Computer Engineering,
Purdue University, West Lafayette, IN 47907, USA.
{zhang70,jianghai@purdue.edu}

**Abstract.** This paper studies the discrete-time linear quadratic regulation problem for switched linear systems (DLQRS) based on dynamic programming approach. The unique contribution of this paper is the analytical characterizations of both the value function and the optimal control strategies for the DLQRS problem. Based on the particular structures of these analytical expressions, an efficient algorithm suitable for solving an arbitrary DLQRS problem is proposed. Simulation results indicate that the proposed algorithm can solve randomly generated DLQRS problems with very low computational complexity. The theoretical analysis in this paper can significantly simplify the computation of the optimal strategy, making an NP hard problem numerically tractable.

## 1 Introduction

A switched system usually consists of a family of subsystems described by differential or difference equations and a logical rule that orchestrates the switching among them. Such systems arise in many engineering fields, such as power electronics [1, 2], embedded systems [3, 4], manufacturing [5], and communication networks [6], etc. In the last decade or so, the stability and stabilizability of switched systems have been extensively studied [7–9]. Many theoretical and numerical tools have been developed for the stability analysis of various switched systems. These stability results have also led to some controller synthesis algorithms which stabilize certain simple switched systems [10]. However, for many engineering applications, ensuring the stability is only the first step rather than an ultimate design goal. How to design a control strategy that not only stabilizes a given switched system, but also optimizes certain design criteria is an even more meaningful yet challenging research problem.

The focus of this paper is on the optimal discrete-time linear quadratic regulation problem for switched linear systems, hereby referred to as the DLQRS problem. The goal is to develop a computationally appealing algorithm to construct an optimal control law that minimizes the given quadratic cost function. The problem is of fundamental importance in both theory and practice and has

* This work was partially supported by the National Science Foundation under Grant CNS-0643805

challenged researchers for many years. The bottleneck is mostly on the determination of the optimal switching strategy. Many methods have been proposed to tackle this problem, most of which are in a divide-and-conquer manner. Algorithms for optimizing the switching instants with a fixed mode sequence have been derived for general switched systems in [11] and for autonomous switched systems in [12]. Although an algorithm for updating the switching sequence is discussed in [12], finding the best switching sequence is still an NP-hard problem, even for switched linear systems.

This paper studies the DLQRS problem from the dynamic programming (DP) perspective. The last few years have seen increasing interest in using DP to solve various optimal control problems of switched systems. In [13], DP is used to derive a search algorithm to find the optimal switching instants for fixed switching sequences. In [14–16], DP-based numerical methods are proposed to compute the optimal switching regions. More recently, Lincoln and Rantzer [17] develop an iterative algorithm to approximate the true value functions with guaranteed accuracy. The algorithm is also used to study switched systems in [18, 19]. Compared with previous studies, the contributions of this paper are the following. First, we characterize analytically the value function and the optimal control strategy for general DLQRS problems. More specifically, we show that the value function at each time step of any DLQRS problem is the pointwise minimum of a finite number of quadratic functions, and that the optimal state-feedback gain is of a Kalman-type form with a state-dependent positive semi-definite matrix. Although other researchers have also suggested a piecewise affine structure for the optimal feedback control [14–16], few of them derive explicitly the optimal feedback gains and identify their connections with the Kalman gain and Riccati recursion of the traditional LQR problem as we do in this paper. Secondly, we prove that under certain conditions of the subsystems, the value function converges exponentially fast as the control horizon increases. Finally, based on the particular structure of the value function and its convergence property, an efficient algorithm is proposed to solve general DLQRS problems. Simulation results indicate that the proposed algorithm can compute the optimal switching strategy and the optimal control input simultaneously with very low computational complexity for randomly generated DLQRS problems. It is worth mentioning that in [17], Lincoln *et al.* proposes a similar structure of the value function when they apply their general theory of relaxed dynamic programming to switched linear systems. The approach adopted in this paper follows naturally from the traditional LQR problem and is substantially different from the one used in [17]. Moreover, different from [17], we allow nonzero terminal cost in the objective function, which is especially important when the time horizon is finite. More comparisons of our result with [17] can be found in Remark 4.

This paper is organized as follows. In Section 2, the DLQRS problem is formulated. The value function of the DLQRS problem is derived in a simple analytical form in Section 3. An algorithm is developed in Sections 4 and 5 to compute the value function in an efficient way. Numerical simulations are performed in Section 6 to demonstrate the algorithm. Finally, some concluding remarks are given in Section 7.

## 2 Problem Formulation

Consider the discrete-time switched linear system defined as:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t),\ t = 0, \dots, N-1, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \dots, M\}$ is the discrete control or switching strategy, and $u(t) \in \mathbb{R}^p$ is the continuous control. For each $i \in \mathbb{M}$, $A_i$ and $B_i$ are constant matrices of appropriate dimension, and the pair $(A_i, B_i)$ is called a subsystem of (1). This switched linear system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^M$ is independent of time $t$. We assume that there is no internal forced switchings, i.e., the system can stay at or switch to any mode at any time instant. In this paper, the terminal cost function $\psi(x)$ and the running cost function $L(x, u, v)$ are assumed to be in the following quadratic forms:

$$\psi(x) = x^T Q_f x, \quad L(x, u, v) = x^T Q_v x + u^T R_v u,$$

where $Q_f = Q_f^T \succeq 0$ is the terminal state weight, and $Q_v = Q_v^T \succeq 0$ and $R_v = R_v^T \succ 0$ are the running weights for the state and the control for subsystem $v \in \mathbb{M}$, respectively. The overall objective function to be minimized over the time horizon $[0, N]$ can thus be defined as

$$J(u, v) = \psi(x(N)) + \sum_{j=0}^{N-1} L(x(j), u(j), v(j))). \tag{2}$$

The goal of this paper is to solve the following discrete-time LQR problem for the switched linear system (1) (referred to as DLQRS problem hereby).

*Problem 1 (DLQRS problem).* Find the $u$ and $v$ that minimize $J(u, v)$ subject to the dynamic equation (1).

## 3 The Value Function of the DLQRS Problem

Following the idea of dynamic programming, for each time $t \in \{0, 1, \dots, N\}$, we define the value function $V_{t,N} : \mathbb{R}^n \to \mathbb{R}$ as:

$$V_{t,N}(z) = \min_{\substack{v(j)\in\mathbb{M},u(j),\\ t\le j\le N-1}} \Big\{ \psi(x(N)) + \sum_{j=t}^{N-1} L(x(j), u(j), v(j)) \Big| \text{ subject to eq. (1) with } x(t) = z \Big\}. \tag{3}$$

The $V_{t,N}(z)$ so defined is the minimum cost-to-go starting from state $z$ at time $t$. The minimum cost for the DLQRS problem with a given initial condition $x(0) = x_0$ is simply $V_{0,N}(x_0)$. Due to the time-invariant nature of the switched system (1), its value function depends only on the number of remaining time steps, i.e.,

$$V_{t,N}(z) = V_{t+m,N+m}(z),$$

for all $z \in \mathbb{R}^n$ and all integers $m \geq -t$. In the rest of this paper, when no ambiguity arises, we will denote by $V_k(z)$ the value function at time $t = N - k$ when there are $k$ time steps left, i.e., $V_k(z) \triangleq V_{N-k,N}(z)$.

In the special case when $M = 1$, the switched system consists of only one subsystem, say, $(A, B)$. Thus, the DLQRS problem degenerates into the classical LQR problem. Denote by $Q$ and $R$ the state and control weighting matrices in this degenerate case. Then, according to the LQR theory, the value function defined in (3) is of the following quadratic form:

$$V_k(z) = z^T P_k z, \quad k = 0, \ldots, N, \tag{4}$$

where $\{P_k\}_{k=0}^N$ is a sequence of positive semi-definite matrices satisfying the Difference Riccati Equation (DRE)

$$P_{k+1} = Q + A^T P_k A - A^T P_k B(R + B^T P_k B)^{-1} B^T P_k A, \tag{5}$$

with initial condition $P_0 = Q_f$. Some important facts about the matrices $P_k$'s are summarized in the following lemma.

**Lemma 1 ( [20, 21]).** *Let $\mathcal{A}$ be the set of all positive semi-definite (p.s.d.) matrices, then*

1. *If $P_k \in \mathcal{A}$, then $P_{k+1} \in \mathcal{A}$.*
2. *If $(A, B)$ is stabilizable, then the sequence $\{||P_k||_2\}_{k=0}^\infty$ is uniformly bounded.*
3. *Let $C$ be a matrix such that $Q = C^T C$. If $(A, B)$ is stabilizable and $(A, C)$ is detectable, then $\lim_{k\to\infty} P_k = P^*$, where $P^*$ is the unique stabilizing solution to the Algebraic Riccati Equation (ARE)*

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA.$$

In general, when $M \geq 2$, the value function $V_k(z)$ is no longer a simple quadratic form as in (4). To derive the value function for the general switched linear system (1), define the Riccati mapping $\rho_i : \mathcal{A} \to \mathcal{A}$ for each subsystem $(A_i, B_i)$, $i \in \mathbb{M}$:

$$\rho_i(P) = Q_i + A_i^T PA_i - A_i^T PB_i(R_i + B_i^T PB_i)^{-1} B_i^T PA_i. \tag{6}$$

Let $\mathcal{H}_0 = \{Q_f\}$ be a set consisting of only one matrix $Q_f$. Define the set $\mathcal{H}_k$ for $k \geq 0$ iteratively as

$$\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k) \triangleq \{P \in \mathcal{A} : P = \rho_i(P_k), \text{for some } i \in \mathbb{M} \text{ and } P_k \in \mathcal{H}_k\}. \tag{7}$$

In other words, each matrix in $\rho_{\mathbb{M}}(\mathcal{H}_k)$ is obtained by taking the Riccati mapping for some matrix in $\mathcal{H}_k$ through some subsystem $i$. Denote by $|\mathcal{H}_k|$ the number of distinct matrices in $\mathcal{H}_k$. Then it can be easily seen that $|\mathcal{H}_0| = 1$ and $|\mathcal{H}_k| \leq M^k$ for any $k \geq 0$.

**Theorem 1.** *The value function for the DLQRS problem at time $N-k$, i.e., with $k$ time steps left, is*

$$V_k(z) = \min_{P \in \mathcal{H}_k} z^T P z. \tag{8}$$

*Furthermore, for $k \geq 0$, if we define*

$$(P_k^*(z), i_k^*(z)) = \underset{(P \in \mathcal{H}_k, i \in \mathbb{M})}{\arg\min}\ z^T \rho_i(P) z, \tag{9}$$

*then the optimal mode (discrete control) and the optimal continuous control at state $z$ and time $N-(k+1)$ are $v^*(N-(k+1)) = i_k^*(z)$ and $u^*(N-(k+1)) = -K_{i_k^*(z)}(P_k^*(z))z$, respectively, where $K_i(P)$ is the Kalman gain for subsystem $i$ with matrix $P$, i.e.,*

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \tag{10}$$

.

*Proof.* The theorem can be proved through induction. It is obvious that for $k = 0$ the value function is $V_k(z) = z^T Q_f z$, satisfying (8). Now suppose equation (8) holds for a general integer $k$, i.e., $V_k(z) = \min_{P \in \mathcal{H}_k} z^T P z$, we shall show that it is also true for $k+1$. By the principle of dynamic programming and noting that $V_k(\cdot)$ represents the value function at time $N-k$, the value function at time $N-(k+1)$ can be recursively computed as

$$\begin{aligned}
&V_{k+1}(z) = \min_{i \in \mathbb{M}, u} \left[ z^T Q_i z + u^T R_i u + V_k(A_i z + B_i u) \right] \\
&= \min_{i \in \mathbb{M}, u} \left[ z^T Q_i z + u^T R_i u + \min_{P \in \mathcal{H}_k} \Big( (A_i z + B_i u)^T P (A_i z + B_i u) \Big) \right] \\
&= \min_{i \in \mathbb{M}, P \in \mathcal{H}_k, u} \left[ z^T Q_i z + u^T R_i u + (A_i z + B_i u)^T P (A_i z + B_i u) \right] \\
&= \min_{i \in \mathbb{M}, P \in \mathcal{H}_k, u} \left[ z^T (Q_i + A_i^T P A_i) z + u^T (R_i + B_i^T P B_i) u + 2 z^T A_i^T P B_i u \right] \\
&\triangleq \min_{i \in \mathbb{M}, P \in \mathcal{H}_k, u} f(i, P, u).
\end{aligned} \tag{11}$$

With a symmetric matrix $P$, it can be easily computed that

$$\frac{\partial f(i, P, u)}{\partial u} = 2(R_i + B_i^T P B_i) u + 2 B_i^T P A_i z.$$

Since $u$ is unconstrained, its optimal value $u^*$ must satisfy $\frac{\partial f(i,P,u^*)}{\partial u} = 0$, i.e.,

$$u^* = -(R_i + B_i^T P B_i)^{-1} B_i^T P A_i z = -K_i(P) z, \tag{12}$$

where $K_i(P)$ is the matrix defined in (10). Substitute $u^*$ into (11), we obtain

$$\begin{aligned} V_{k+1}(z) &= \min_{i\in\mathbb{M}, P\in\mathcal{H}_k} f(i,P,u^*) \\ &= \min_{i\in\mathbb{M}, P\in\mathcal{H}_k} \left[ z^T \Big( Q_i + A_i^T P A_i - A_i^T P B_i K_i(P) \Big) z \right] \\ &= \min_{i\in\mathbb{M}, P\in\mathcal{H}_k} z^T \rho_i(P) z. \end{aligned}$$

Let $P_k^*(z)$ and $i_k^*(z)$ be the matrix and the index that minimize $z^T\rho_i(P)z$, i.e., they are defined as in (9). Then the optimal continuous control and discrete control at time $N-(k+1)$ and state $z$ are $u^*(N-(k+1)) = -K_{i_k^*(z)}(P_k^*(z))z$ and $v^*(N-(k+1)) = i_k^*(z)$, respectively. Furthermore, observing that $\{\rho_i(P) : i \in \mathbb{M}, P \in \mathcal{H}_k\} = \rho_{\mathbb{M}}(\mathcal{H}_k) = \mathcal{H}_{k+1}$, we have $V_{k+1}(z) = \min_{P\in\mathcal{H}_{k+1}} z^T P z$.

According to Theorem 1, comparing to the discrete-time LQR case, the value function of the DLQRS problem is no longer a single quadratic function; it actually becomes the pointwise minimum of a finite number of quadratic functions. In addition, at each time step, instead of having a single Kalman gain for the entire state space, the optimal state feedback gain becomes state dependent. Furthermore, the minimizer $(P_k^*(z), i_k^*(z))$ of equation (9) is radially invariant, indicating that at each time step all the states along the same radial direction have the same optimal mode and optimal feedback gain.

## 4 Equivalent Subset of p.s.d. Matrices

According to Theorem 1, the value function $V_k(\cdot)$ is completely characterized by the set $\mathcal{H}_k$, which can be obtained iteratively by (7). Since the size of the set $\mathcal{H}_k$ grows exponentially fast, it becomes unfeasible to compute $\mathcal{H}_k$ when $k$ gets large. However, in terms of computing the value function, we only need to keep the matrices in $\mathcal{H}_k$ that give rise to the minimum of (8) for at least one $z \in \mathbb{R}^n$. To remove the redundant matrices in $\mathcal{H}_k$ and simplify the compuation, the following definitions are introduced.

**Definition 1 (Equivalent Sets of p.s.d Matrices).** *Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be two sets of p.s.d matrices. The set $\mathcal{H}$ is called* equivalent *to $\hat{\mathcal{H}}$, denoted by $\mathcal{H} \sim \hat{\mathcal{H}}$, if $\min_{P\in\mathcal{H}} z^T P z = \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$, $\forall z \in \mathbb{R}^n$.*

Therefore, any equivalent sets of p.s.d. matrices will define the same value function of the DLQRS problem. To ease the computation, we are more interested in finding the smallest equivalent subset of $\mathcal{H}_k$.

**Definition 2 (Minimum Equivalent Subset (MES)).** *Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be two sets of symmetric p.s.d matrices. $\hat{\mathcal{H}}$ is called an* equivalent subset *of $\mathcal{H}$ if $\hat{\mathcal{H}} \subseteq \mathcal{H}$ and $\hat{\mathcal{H}} \sim \mathcal{H}$. Furthermore, $\hat{\mathcal{H}}$ is called a* minimum equivalent subset *(MES) of $\mathcal{H}$ if it is the equivalent subset of $\mathcal{H}$ with the fewest elements. Note that the MES of $\mathcal{H}$ may not be unique. Denote by $\Gamma(\mathcal{H})$ one of the MES's of $\mathcal{H}$.*

*Remark 1.* It is also worth mentioning that due to its special structure, the value function is homogeneous, namely, $V_k(\lambda z) = \lambda^2 V_k(z)$, for all $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^1$. Therefore, it suffices to consider only the points $z$ on the unit sphere in checking the conditions in the above two definitions.

The following lemma provides a test for the equivalent subsets of $\mathcal{H}_k$.

**Lemma 2.** *$\hat{\mathcal{H}}$ is an equivalent subset of $\mathcal{H}$ if and only if*

1. *$\hat{\mathcal{H}} \subseteq \mathcal{H}$*
2. *$\forall P \in \mathcal{H}$ and $\forall z \in \mathbb{R}^n$, there exists a $\hat{P} \in \hat{\mathcal{H}}$ such that $z^T \hat{P} z \leq z^T P z$.*

*Proof.* (a) (sufficiency): We need to prove $\min_{P\in\mathcal{H}} z^T P z = \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$, $\forall z \in \mathbb{R}^n$. Obviously $\min_{P\in\mathcal{H}} z^T P z \leq \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$, $\forall z \in \mathbb{R}^n$ because $\hat{\mathcal{H}} \subseteq \mathcal{H}$. On the other hand, by the second condition, for each $z \in \mathbb{R}^n$ and $P \in \mathcal{H}$, there exist a $\hat{P}$ such that $z^T \hat{P} z \leq z^T P z$. Thus, $\min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z \leq \min_{P\in\mathcal{H}} z^T P z$. (b) (necessity): straightforward by a standard contradiction argument.

*Remark 2.* Lemma 2 can be used as an alternative definition of the equivalent subset. Although the original definition is conceptually simpler, the conditions given in this lemma provide a more explicit characterization of the equivalent subset, which proves to be more beneficial in the subsequent discussions.

All the equivalent subsets of $\mathcal{H}_k$ define the same value function $V_k(z)$. Thus, in terms of computing the value function, all the matrices in $\mathcal{H}_k \setminus \Gamma(\mathcal{H}_k)$ are redundent. More rigorously, $\hat{P} \in \mathcal{H}_k$ is called *redundent* with respect to $\mathcal{H}_k$ if for all $z \in \mathbb{R}^n$, there exists a $P \in \mathcal{H}_k \setminus \{\hat{P}\}$ such that $z^T \hat{P} z \geq z^T P z$. Thus, to simplify the computation, we shall prune away as many as possible redundant matrices and obtain an equivalent subset of $\mathcal{H}_k$ as close as possible to $\Gamma(\mathcal{H}_k)$. However, testing whether a matrix is redundant or not is itself a challenging problem. Geometrically, any p.s.d. matrix defines uniquely an ellipsoid in $\mathbb{R}^n$. It can be easily verified that $\hat{P} \in \mathcal{H}_k$ is redundant if and only if its corresponding ellipsoid is completely contained in the union of all the ellipsoids corresponding to the matrices in $\mathcal{H}_k \setminus \{\hat{P}\}$. Since the union of ellipsoids are not convex in general, there is in general no efficient way to verify this geometric condition or equivalently the original mathematical condition of redundancy. Nevertheless, a sufficient condition for a matrix to be redundant can be easily obtained and is given in the following lemma.

**Lemma 3.** *$\hat{P}$ is redundant with respect to $\mathcal{H}_k$ if there exist nonnegative constants $\alpha_1, \ldots, \alpha_{|\mathcal{H}_k|-1}$ such that $\sum_{i=1}^{|\mathcal{H}_k|-1} \alpha_i = 1$ and $\hat{P} \succeq \sum_{i=1}^{|\mathcal{H}_k|-1} \alpha_i P^{(i)}$, where $\{P^{(j)}\}_{j=1}^{|\mathcal{H}_k|-1}$ is an enumeration of $\mathcal{H}_k \setminus \{\hat{P}\}$.*

*Proof.* Straightforward.

For given $\hat{P}$ and $\mathcal{H}_k$, the condition in Lemma 3 can be easily verified using various existing convex optimization algorithms [22]. Lemma 3 can not guarantee to identify all the redundent matrices, however, it usually can help to eliminate

**Algorithm 1**

1. Denote by $P^{(j)}$ the $j^{(th)}$ matrix in $\mathcal{H}_k$. Set $\mathcal{H}_k^{(1)} = \{P^{(1)}\}$.
2. For each $j = 2, \ldots, |\mathcal{H}_k|$, if $P^{(j)}$ satisfies the condition in Lemma 2 with respect to $\mathcal{H}_k$, then $\mathcal{H}_k^{(j)} = \mathcal{H}_k^{(j-1)}$, otherwise $\mathcal{H}_k^{(j)} = \mathcal{H}_k^{(j-1)} \cup \{P^{(j)}\}$.
3. Return $\mathcal{H}_k^{(|\mathcal{H}_k|)}$.

a large portion of the redundant matrices in $\mathcal{H}_k$. During the value iteration, each matrix in $\mathcal{H}_k$ will be tested according to Lemma 3. If the condition in Lemma 3 is met, then the matrix under consideration will be discarded; otherwise, the matrix will be kept and used to generate the set $\mathcal{H}_{k+1}$. A detailed description of this process is given in Algorithm 1. The returned set $\mathcal{H}_k^{(|\mathcal{H}_k|)}$ by Algorithm 1 is an equivalent subset of $\mathcal{H}_k$ with usually a much smaller size.

## 5 Computation of the Value Function

In this section, we use the equivalent-subset concept to simplify the computation of the value function of the DLQRS problem. For each $k \leq N$, let $\hat{\mathcal{H}}_k$ be an arbitrary equivalent subset of $\mathcal{H}_k$. The following corollary follows immediately from Definition 2.

**Corollary 1.** *The result in Theorem 1 still holds if every $\mathcal{H}_k$ is replaced by $\hat{\mathcal{H}}_k$.*

Corrollary 1 says that to compute the value function and the optimal control strategy, it suffices to use an equivalent subset of $\mathcal{H}_k$ for each $k$. In the last section, we have developed an algorithm to prune the redundant matrices in $\mathcal{H}_k$. However, the complexity of the algorithm still depends on $|\mathcal{H}_k|$, which grows exponentially fast as $k$ increases. To overcome this difficulty, the following lemma is introduced.

**Lemma 4 (Self Iteration).** *Let the sequence of sets $\{\hat{\mathcal{H}}_k\}_{k=0}^N$ be generated by*

$$\hat{\mathcal{H}}_0 = \mathcal{H}_0, \text{ and } \hat{\mathcal{H}}_{k+1} = Algo(\rho_{\mathbb{M}}(\hat{\mathcal{H}}_k)) \text{ for } 0 \leq k \leq N-1, \tag{13}$$

*where $Algo(\mathcal{H})$ denotes the equivalent subset of $\mathcal{H}$ returned by Algorithm 1. Then $\hat{\mathcal{H}}_k \sim Algo(\mathcal{H}_k)$.*

*Proof.* The interested readers are refered to [23] for the proof of this lemma.

According to Lemma 4, $Algo(\mathcal{H}_k)$ is equivalent to $Algo(\rho_{\mathbb{M}}(\hat{\mathcal{H}}_{k-1}))$. Thus, to compute the desired equivalent subset of $\mathcal{H}_k$, one can apply Algorithm 1 to $\rho_{\mathbb{M}}(\hat{\mathcal{H}}_{k-1})$ instead of the original set $\mathcal{H}_k$. Denoted by $|\hat{\mathcal{H}}_k|$ the size of $\hat{\mathcal{H}}_k$. The set $\rho_{\mathbb{M}}(\hat{\mathcal{H}}_{k-1})$ contains at most $M \cdot |\hat{\mathcal{H}}_k|$ matrices which is usually much smaller than $|\mathcal{H}_k| = M^k$. Therefore, Lemma 2 could significantly simplify the computation of $Algo(\mathcal{H}_k)$. Although $|\hat{\mathcal{H}}_k|$ grows reasonably slow, it is still possible to become out of hand if the control horizon $N$ is large. The following theorem allows us to terminate the computation with guaranteed accuracy on the optimal cost at some early stage for large time horizon $N$.

**Theorem 2.** *Suppose that (i) $Q_f \succ 0$ and $Q_i \succ 0$ for each $i \in \mathbb{M}$; (ii) at least one subsystem is stabilizable. Then $V_k(z)$ converges exponentially fast to $V_\infty(z)$ for each $z \in \mathbb{R}^n$ as $k \to \infty$. Furthermore, the convergence is uniform on the unit sphere in $\mathbb{R}^n$ and the difference between the value functions at time step $N - k_1$ and $N - k_2$ is bounded above by*

$$|V_{k_1}(z) - V_{k_2}(z)| \leq (\beta + \lambda_f^+)\alpha\gamma^{k_2}\|z\|^2, \tag{14}$$

*where $\beta$, $\lambda_f^+$, $\alpha$ and $\gamma$ are all parameters depending only on the subsystem matrices.*

*Remark 3.* Note that the first condition in the above theorem is not restrictive because a randomly selected p.s.d. matrix is almost surely nonsingular. The proof of this theorem is quite involved and is beyond the scope of this paper. The interested readers are referred to [23] for a complete proof.

*Remark 4.* Compared with the convergence result in [17], Theorem 2 has several distinctive features. Firstly, it allows nonzero terminal cost, which is especially important for finite-horizon DLQRS problem. Secondly, its conditions are much easier to verify as they are expressed in terms of the system matrices instead of the infinite-horizon value functions as is the case in [17]. Finally, by inequality (14), the convergence rate can be approximated using the system matrices. Thus, for a given tolerance on the optimal cost, an upper bound of the required number of iterations can be simply computed before the actual computation starts. This provides an efficient means to stop the value iterations.

The exponential convergence result is crucial for the efficient computation of the value function. Given a reasonable tolerance on the accuracy, the value function usually converges in only a few steps. This greatly simplifies the value function computation, especially for the case with large time horizon $N$. In practice the convergence is usually tested only on a finite set of sampling points on the unit sphere. These sampling points should be chosen dense enough to capture the behaviors of all the value functions on the entire unit sphere. The existence of such sampling points is guaranteed by the following corollary of Theorem 2.

**Corollary 2.** *Under the same conditions as in Theorem 2, the sequence of value functions $\{V_k(z)\}_{k=0}^{\infty}$ is equicontinuous on the unit sphere.*

*Proof.* Denote by $B_u$ the unit sphere in $\mathbb{R}^n$. Obviously, each value function $V_k(z)$ is continuous on $B_u$. By theorem 2, $V_k(\cdot)$ converges uniformly on $B_u$. Since $B_u$ is a compact set, the desired result follows directly from Theorem 7.24 in [24].

With all the results developed so far, a general procedure for solving the DLQRS problem is summarized in Algorithm 2.

**Algorithm 2**

1. Set $\hat{\mathcal{H}}_0 = Q_f$ and specify a tolerance $\epsilon$ for the minimum cost. Choose a finite set of sampling points on the unit sphere of $R^n$ and denote it by $\mathbb{S}$.
2. For each step $k \geq 1$, compute $\hat{\mathcal{H}}_k = Algo(\rho_{\mathbb{M}}(\hat{\mathcal{H}}_{k-1}))$ where $Algo(\cdot)$ represents Algorithm 1.
3. Compute the value function $V_k(z)$ for each $z \in \mathbb{S}$ using $\hat{\mathcal{H}}_k$.
4. If $|V_k(z) - V_{k-1}(z)| > \epsilon$ for some $z \in \mathbb{S}$, then let $k = k+1$ and go back to step 2. Otherwise let $k_\epsilon = k$ and continue to step 5.
5. Define $\hat{\mathcal{H}}_k = \hat{\mathcal{H}}_{k_\epsilon}$ for $k_\epsilon \leq k \leq N$.
6. The optimal trajectory can now be obtained by

$$x(t+1) = A_{v^*(t)}x(t) + B_{v^*(t)}u^*(t), \text{ with } x(0) = x_0,$$

where $v^*(t)$ and $u^*(t)$ are determined using Corollary 1 based on the set $\hat{\mathcal{H}}_{N-(t+1)}$.

**Table 1.** $|\hat{\mathcal{H}}_k|$ for Ex1

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\lvert\hat{\mathcal{H}}_k\rvert$ | 2 | 4 | 5 | 5 | 5 | 5 |

# 6 Examples

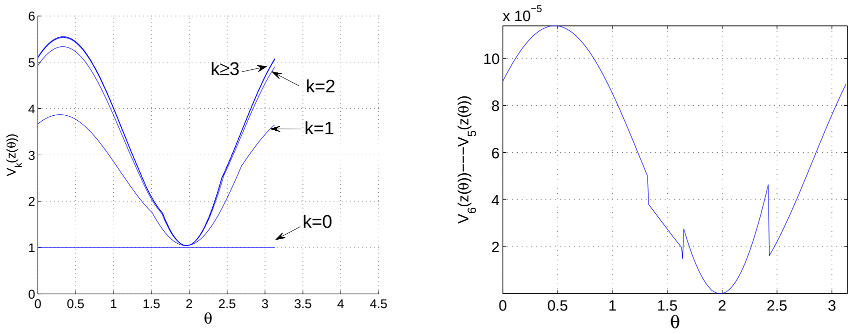
## 6.1 Example 1

First consider a simple DLQRS problem, referred to as Ex1, with control horizon $N = 100$ and two second-order subsystems:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Suppose that state and control weights are $Q_1 = Q_2 = I_{2\times2}$ and $R_1 = R_2 = 1$, respectively. Both subsystems are unstable but controllable. Algorithm 2 is applied to solve this DLQRS problem. It turns out that with the error tolerance $\epsilon = 10^{-3}$ the value function of Ex1 converges in 6 steps. Since $V_k(z)$ is homogeneous, it suffices to plot it at the points on the unit circle, i.e. the points of the form $z(\theta) = [\cos(\theta), \sin(\theta)]^T$. It can be easily verified that $V_k(z(\theta)) = V_k(z(\theta + \pi))$, i.e., the value function is periodic along the unit circle with period $\pi$. Therefore, in Fig. 1-(a), the value function at each time step is plotted only at the points $z(\theta)$ with $\theta \in [0, \pi]$. The difference between the value functions in the last two iterations are shown in Fig. 1-(b). The number of elements in $\hat{\mathcal{H}}_k$ at each step is listed in Table 1. It can be seen that $|\hat{\mathcal{H}}_k|$ is indeed very small, and will stay at the maximum value 5 as opposed to growing exponentially as k increases.

Furthermore, the optimal switching strategy is illustrated in Fig. 2. At each time step, the whole state space is divided into several conic regions. The regions




(a) (b)

**Fig. 1.** Convergence results for Ex1. (a) Convergence of the Value function. (b) Difference between the last two iterations.

with the same gray scale have the same optimal mode. However, the points with the same optimal mode may correspond to different optimal feedback gains. The radial lines in Fig 2 further divide the optimal-mode regions into smaller conic regions each with a different optimal-feedback gain. In this way, the proposed approach actually characterizes the optimal control strategies for the entire state space.

**Table 2.** $|\hat{\mathcal{H}}_k|$ for Ex2

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\lvert\hat{\mathcal{H}}_k\rvert$ | 3 | 9 | 15 | 15 | 15 |

### 6.2 Example 2

Consider a more complex DLQRS example, referred to as Ex2, with 4 subsystems. The first two subsystems are the same as in Ex1 and the other two are defined as:

$$A_3 = \begin{bmatrix} 3 & 1 \\ 0 & 0.2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix}, \quad B_3 = B_1, \quad \text{and} \quad B_4 = B_2.$$

With the same tolerance, the value function of Ex2 converges in 5 steps. This indicates that under the same tolerance, the speed of the convergence of the value function may not necessarily increase with the number of subsystems. However, with more subsystems, $|\hat{\mathcal{H}}_k|$ grows more rapidly as shown in Table 2. It is worth mentioning that the maximum $|\hat{\mathcal{H}}_k|$ for this example is only 15 (as opposed to the nominal size of $\mathcal{H}_N$, $|\mathcal{H}_N| = 4^{100}$). Therefore, the proposed method has dramatically simplified the problem, making an NP hard problem numerically tractable.

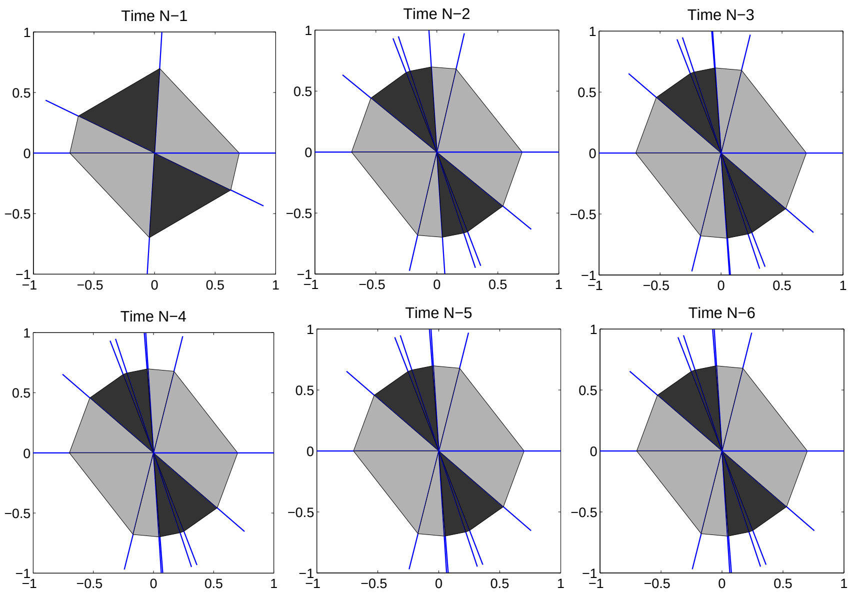



**Fig. 2.** Switching Regions for Ex1: Gray Region – mode 1is optimal; Black Region – mode 2 is optimal.

### 6.3 Random Examples

This subsection is devoted to demonstrating the generic solvability of a general second-order DLQRS problem using the proposed algorithm. Our goal here is not to present a formal proof but rather to illustrate through simulations some important observations. In this set of simulations, the proposed algorithm is tested on 1000 randomly generated second-order DLQRS problems with a fairly large number of subsystems ($M = 10$). The control horizon is the same as in the last two examples, i.e., $N = 100$. All of these problems are successfully solved and the distribution of $|\hat{\mathcal{H}}_{k_\epsilon}|$, namely, the maximum number of matrices kept before convergence, is plotted in Fig. 3. It can be seen from the figure that the number $|\hat{\mathcal{H}}_{k_\epsilon}|$ in all of these 1000 problems are smaller than 50, and for a majority of the problems, $|\hat{\mathcal{H}}_{k_\epsilon}|$ is smaller than 15. Therefore, most second-order DLQRS problems may be efficiently solved using the proposed algorithm. Formally proving the generic solvability is a focus of our future research.

## 7 Conclusion

This paper studies the DLQRS problem based on dynamic programming approach. Different from the traditional LQR problem, the value function of the DLQRS problem is no longer a single quadratic function; it is the pointwise minimum of a finite number of quadratic functions. In addition, instead of having

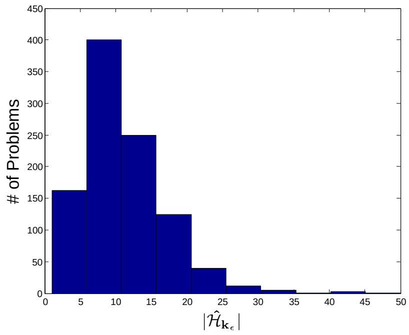



**Fig. 3.** Distribution of $|\hat{\mathcal{H}}_{k_\epsilon}|$ for randomly generated problems

a single Kalman feedback gain as in the LQR case, the optimal state-feedback gain in the DLQRS problem becomes state dependent. Analytical expressions have been derived for both the optimal switching strategy and optimal control inputs. The concept of minimum equivalent subsets is introduced to simplify the computation of the value function. An efficient algorithm is developed to compute the optimal control strategy with guaranteed accuracy on the optimal cost. Simulation results indicate that the proposed algorithm can efficiently solve any randomly generated second-order DLQRS problems. Future research will focus on how to compute the exact MES of $\mathcal{H}_k$ in a general-dimensional state space and on proving the generic solvability of general DLQRS problems.

## References


1. T. Geyer, G. Papafotiou, and M. Morari. Model Predictive Control in Power Electronics: A Hybrid Systems Approach. In *IEEE Conference on Decision and Control*, Seville, Spain, December 2005.
2. A.G. Beccuti, G. Papafotiou, and M. Morari. Optimal Control of the Boost dc-dc Converter. In *IEEE Conference on Decision and Control*, Seville, Spain, December 2005.
3. W. Zhang, J. Hu, and Y.-H. Lu. Optimal Power Modes Scheduling Using Hybrid Systems. In *Proceedings of the American Control Conference*, New York City, NY, July 2007.
4. W. Zhang and J. Hu. Optimal Buffer Management Using Hybrid Systems. In *IEEE Conference on Decision and Control*, New Orleans, LA, December 2007.
5. C. G. Cassandras, D. L. Pepyne, and Y. Wardi. Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, 46(3):398–415, 2001.
6. J. Hespanha, S. Bohacek, K. Obraczka, and J. Lee. Hybrid modeling of TCP congestion control. In Maria Domenica Di Benedetto and Alberto Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control*, number 2034 in LNCS, pages 291–304. Springer-Verlag, Berlin, March 2001.
7. D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

8. D. Liberzon, J. P. Hespanha, and A. S. Morse. Stability of switched systems: a lie-algebraic condition. *Systems and Control Letters*, 37(3):117–122, 1999.
9. R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of IEEE, Special Issue on Hybrid Systems*, 88(7):1069–1082, 2000.
10. S. Pettersson. Synthesis of switched linear systems. In *IEEE Conference on Decision and Control*, pages 5283– 5288, Maui, HI, Dec 2003.
11. X. Xu and P.J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Transactions on Automatic Control*, 49(1):2–16, 2004.
12. M. Egerstedt, Y. Wardi, and F. Delmotte. Optimal control of switching times in switched dynamical systems. *IEEE Transactions on Automatic Control*, 51(1):110–115, 2006.
13. X. Xu and P. Antsaklis. A dynamic programming approach for optimal control of switched systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1822–1827, Sydney, Australia, December 2000.
14. A. Bemporad, A. Giua, and C. Seatzu. Synthesis of state-feedback optimal controllers for continuous-time switched linear systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3182– 3187, 2002.
15. F. Borrelli, M. Baotic, A. Bemporad, and M. Morari. Dynamic programming for constrained optimal control of discrete-time linear hybrid systems. *Automatica*, 41:1709–1721, October 2005.
16. C. Seatzu, D. Corona, A. Giua, and A. Bemporad. Optimal control of continuous-time switched affine systems. *IEEE Transactions on Automatic Control*, 51:726–741, May 2006.
17. B. Lincoln and A. Rantzer. Relaxing dynamic programming. *IEEE Transactions on Automatic Control*, 51(8):1249–1260, August 2006.
18. A. Rantzer. Relaxed dynamic programming in switching systems. *IEE Proceedings - Control Theory & Applications*, 153(5):567 – 574, September 2006.
19. M. Rinehart, M. Dahleh, and I. Kolmanovsky. Optimal control of switched homogeneous systems. In *Proceedings of the American Control Conference*, pages 1377–1382, New York City, NY, Jul 2007.
20. P. E. Caines and D. Q. Mayne. On the discrete time matrix riccati equation of optimal control. *Internation Journal of Control*, 12(5):785–794, 1970.
21. Siew Chan, Goodwin, and G. Kwai Sin. Convergence properties of the riccati difference equation in optimal filtering of nonstabilizable systems. *IEEE Transactions on Automatic Control*, 29(2):110–118, 1984.
22. S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.
23. W. Zhang and J. Hu. Optimal quadratic regulation for linear switched systems. to be submitted for journal publication (available upon request).
24. R. Walter. *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.