Delay and Slew Metrics Using the Lognormal Distribution

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ABSTRACT
For optimizations like physical synthesis and static timing analysis, efficient interconnect delay and slew computation is critical. Since one cannot afford to run AWE [12], constant time solutions are required. This work presents the first complete solution to closed form formulae for both delay and slew. Our metrics are derived from matching circuit moments to the lognormal distribution. From a single table, one can easily implement the metrics for delay and slew for both step and ramp inputs. Experiments validate the effectiveness of the metrics for nets from a real industrial design.

Categories and Subject Descriptors
B.7.2 [Integrated Circuits]: Design Aids - Simulation;

General Terms
Algorithms, Design, Theory

1. INTRODUCTION
Delay computation is key for both performance estimation and optimization of high performance integrated circuits. Interconnect (or linear circuit) delay computation is at the core of physical analysis and physical design tools. One must be able to efficiently and accurately compute interconnect delay and slew since several million calculations are required to analyze and optimize a design.

By interpreting the impulse response of a linear circuit as a probability distribution function (PDF), Elmore [4] proposed using the mean of the impulse response to approximate the median of the impulse response under the probability interpretation under a step excitation. The Elmore delay metric has been incredibly popular because it is simple, closed-form and easy to evaluate. This metric was resurrected when Rubenstein et al. [13] published a simple closed-form formula for computing the mean of the impulse response of RC interconnect trees. The widely known achilles heel of the Elmore metric is that it is highly inaccurate when there is a high degree of resistive shielding.

The Elmore metric can be improved by computing and matching higher order moments of the impulse response via AWE (Asymptotic Waveform Evaluation) [12]. However, AWE cannot be expressed by a closed-form formula, instead requiring the solution of a non-linear equation. A closed form metric is preferable for both efficiency and implementation simplicity, as long as it is sufficiently accurate. Pileggi [11] has written a survey of timing metrics for RC trees.

Alpert et al. [1] proposed the D2M metric which is a simple function of the first two circuit moments. In this work, we present the first theoretical analysis behind D2M. The PRIMO [8], h-Gamma [9], and WED [10] metrics are based on matching the moments of the impulse response to a PDF. The first two match to the Gamma distribution, while WED matches to the Weibull distribution. All require some type of table lookup operation.

While delay metrics are fairly well-studied, few metrics have been proposed to compute slew. As ultra deep sub-micron effects continue to wreak havoc on signal integrity, computing slew efficiently and accurately has become increasingly critical.

Elmore [4], as it turns out, also proposed a slew metric in his seminal paper, namely a constant times “radius of gyration”, i.e., the standard deviation of the impulse response. Gupta et al. [5] also recognized this as a “good measure” for slew. Bakoglu [3] proposed a first-order slew metric which is a constant times the Elmore delay. Bakoglu’s metric can be derived by matching the first moment of the impulse response to the mean of exponential distribution.

We present closed form metrics based on the lognormal distribution. Unlike [8][9][10], matching to the lognormal distribution produces closed form formulae. We make the following contributions:

• A simple delay metric LnD (lognormal delay) is derived using the first two moments of the impulse response. The resulting delay metric is actually D2M [1] though with a different scaling constant. Hence, the derivation of LnD actually provides the first theoretical justification for the highly accurate D2M (and hence LnD) delay metrics.

• A closed form slew metric LnS (lognormal slew) is presented from a derivation of the first three circuit moments.

• The LnD and LnS metrics can be extended to ramp inputs using the PERI method [7]. The results are consolidated into a single table that makes it easy for the reader to implement the metrics for both step and ramp inputs.

The effectiveness of the lognormal metrics is demonstrated on nets from an industrial design.

2. BACKGROUND
Assume that \( h(t) \) is the impulse response of a node voltage in an RC circuit. The circuit moments of the impulse response are

\[
m_k = \frac{(-1)^k}{k!} \int_0^\infty t^k h(t) dt
\]

The circuit moments can be computed directly as functions of the RC’s in time linear in the size of the circuit, e.g., via path tracing. The impulse response \( h(t) \) satisfies the following conditions [13]:

\[
h(t) \geq 0 \quad \text{and} \quad \int_0^\infty h(t) dt = 1
\]

Consequently, the impulse response is a PDF, though there is no known underlying statistical distribution describing it (which is why fast and accurate delay and slew computation is so difficult).

The mean of the impulse response is:

\[
\mu = \int_0^\infty t \cdot h(t) dt
\]

Elmore [4] showed that \( \mu = -m_1 \) and therefore approximated the median (the desired delay) by the mean of the impulse response. We let \( ED = \mu \) denote the Elmore delay. The \( k^{th} \) central moment is given by

\[
\mu_k = \int_0^\infty (t - \mu)^k h(t) dt
\]

The variance \( (\sigma^2) \) and the skewness \( (\gamma) \) of the impulse response can be expressed in terms of the central moments and also the circuit moments [5]:

\[
\sigma^2 = \mu_2 = 2m_2 - m_1^2
\]

\[
\gamma = \frac{\mu_3}{\sigma^3} = \frac{-6m_3 + 6m_1m_2 - 2m_1^2}{(2m_2 - m_1^2)^{3/2}}
\]

The key idea behind our delay and slew metrics is to match the mean, variance and skewness of the impulse response to those of the lognormal distribution.
3. LOGNORMAL DELAY METRIC

The lognormal distribution $t \sim \text{Logn}(M, S)$ is a two-parameter continuous distribution in which the logarithm of the input variable has a Gaussian distribution. The lognormal distribution is well-suited to match the impulse response since both are unimodal and have nonnegative skewness. The lognormal PDF is given by

$$P(t) = \frac{1}{tS\sqrt{2\pi}} e^{-\frac{(\ln t - M)^2}{2S^2}}$$

where $M > 0$ and $S > 0$ are the scale and shape parameters, respectively. Its cumulative density function (CDF) is given by

$$D(t) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\ln t - M}{S\sqrt{2}} \right) \right).$$

The expected value (or mean) and the variance are respectively given by

$$E(t) = e^{M + S^2/2} \quad \text{and} \quad V(t) = e^{2M + S^2}(e^S - 1).$$

One can match two common properties of the lognormal distribution and the circuit’s impulse response. Recall that the mean and variance of the impulse response are $\mu = m_1$ and $\sigma^2 = 2m_2 - m_1^2$, respectively. Using Equation (9) to match the mean and variance yields the values $m_1 = e^{M + S^2/2}$, $S = e^{2M} - e^{S^2} - 1$.

Solving this system for $M$ and $S$ yields

$$M = \ln(m_1^2/2m_2) \quad \text{and} \quad S = \sqrt{\ln(2m_2/m_1^2)}$$

The median of the lognormal distribution is given by $e^M$. One can verify this by setting $D(t) = 0.5$ in Equation (8) and solving for $t$. Thus, when matching the impulse response the median becomes our 50% delay metric:

$$\text{LnD} = e^{\ln(m_1^2/2m_2)} = m_1^2/2m_2.$$ (12)

Thus, the delay function is a simple function of the first two circuit moments. Observe that this metric is actually similar to the metric [2] $D2M = m_1^2 \ln 2/m_2$. In fact, observe that

$$\text{D2M} = \ln 2 = 0.9802,$$

$\text{LnD} = \ln 2/\sqrt{2}$, (13)

i.e., D2M and LnD are actually the same delay metric, except for a percent constant difference. D2M was derived empirically and shown to be accurate. By matching to the lognormal distribution, we provide the first theoretical justification of the D2M metric.

Using the fact that $2m_2 - m_1^2 \geq 0$, it follows as in [1] that $\text{LnD} \leq m_1$. This, the LnD metric is bounded above by the Elmore delay and is always nonnegative.

4. LOGNORMAL SLEW METRIC

To derive a slew metric for say 10% slew, one can use the same distribution matching as for the delay metric and then compute the 90% and 10% delay points. Note that these formulae can be generalized for other measurement points. Let $t_{lo}$ and $t_{hi}$ be the 10/90 delay points. From Equation (8), the 10% and 90% delay points can be found by setting $D(t_{lo}) = 0.1$ and $D(t_{hi}) = 0.9$, i.e.,

$$0.1 = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\ln t_{lo} - M}{S\sqrt{2}} \right) \right), \quad 0.9 = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\ln t_{hi} - M}{S\sqrt{2}} \right) \right).$$ (14)

or

$$\text{erf} \left( \frac{\ln t_{lo} - M}{S\sqrt{2}} \right) = -0.8 \quad \text{and} \quad \text{erf} \left( \frac{\ln t_{hi} - M}{S\sqrt{2}} \right) = 0.8.$$ (15)

Let $k$ be such that $\text{erf}(k) = 0.8 \quad (k = 0.9062)$. Since $\text{erf}(x) = -\text{erf}(-x)$, solving for $t_{lo}$ and $t_{hi}$ yields:

$$t_{lo} = e^{M - Sk\sqrt{2}} \quad \text{and} \quad t_{hi} = e^{M + Sk\sqrt{2}}.$$ (16)

So the formula for slew becomes

$$\text{LnS} = t_{hi} - t_{lo} = \frac{M}{\sqrt{2}} \left( e^{Sk\sqrt{2}} - e^{-Sk\sqrt{2}} \right).$$ (17)

The only part that remains is to match moments to derive values for the lognormal distribution parameters $M$ and $S$. Using the values from Equation (11) derived from matching the mean and variance yields the following slew metric:

$$\text{LnS}_{12} = \frac{1}{2} \left( e^{\frac{k\sqrt{2}2\ln(2m_2/m_1^2)} - e^{-k\sqrt{2}2\ln(2m_2/m_1^2)}} \right).$$ (18)

We call this metric LnS12 for Lognormal Slew matching the first and second moments. Empirically, LnS12 is fairly accurate at the near-end, but tends to underestimate slew by about 20% at the far-end. One reason is that it ignores the skewness of the impulse response. While the variance describes how wide the waveform spreads (the essential notion of slew), skewness reflects its degree of asymmetry. At the far-end when the impulse response flattens, skewness becomes more important than the mean.

Hence, we now derive an alternative slew metric by matching variance and skewness. The skewness of the lognormal distribution is given by

$$\mu_3 = \frac{\sigma^3}{\mu^3} = \frac{-6m_3 + 6m_1m_2 - 2m_1^3}{(2m_2 - m_1^2)^{3/2}}.$$ (20)

Setting $G(t) = \gamma$ and solving for $S$ yields two complex roots and one real root. The real root is given by $S = \sqrt{\ln(z)}$

where $z = \left( \frac{y - 1}{\gamma} \right)^2 + 1$ and $y = \left( \frac{\gamma + 3 + \gamma^2}{2} \right)^{1/3}$.

Next, we match the variance by setting $V(t)$ in Equation (9) equal to $\sigma^2 = 2m_2 - m_1^2$, yielding:

$$e^{2M + S^2} (e^{S^2} - 1) = 2m_2 - m_1^2.$$ (22)

Solving for $M$ yields:

$$M = \ln \left( \frac{2m_2 - m_1^2}{e^{S^2} - 1} \right) = \ln \left( \frac{2m_2 - m_1^2}{z(z - 1)} \right).$$ (23)

Substituting these values back into Equation (17) yields the LnS23 metric (since it matches the second and third circuit moments):

$$\text{LnS}_{23} = \frac{2m_2 - m_1^3}{z(z - 1)} \left( e^{\frac{2\ln(z)}{2}} - e^{-\frac{2\ln(z)}{2}} \right).$$ (24)

Note that the formula for LnS23 looks like the standard deviation $\sigma$ times a constant that depends only on the skewness of the distribution. One can verify that for a single RC network, $\gamma = 2$. In this case, the constant term times $\sigma$ evaluates to 2.206, which is very close to $\ln(9) = 2.197$, the correct value for when there is only one pole.

Empirically, LnS12 is better suited for the near-end and LnS23 is better suited for the far-end. One might think that LnS23 would also be more appropriate at the near-end, where the skewness of the impulse response is large. However, the large skewness actually reflects the long tail of the waveform, which is often well beyond even the 90% delay point. For these cases, using skewness instead of the mean can introduce large error. For the near end nodes, LnS12 remains a better choice.

To know which metric to use, one must be able to identify whether a sink is near-end or far-end. For this purpose, we use the ratio $r = m_1/\sqrt{m_2}$. This ratio $r$ is typically much smaller than one for a near-end sink and is provably greater than one for a far-end sink. We seek to have a smooth tradeoff between the near-end and far-end via a linear combination of the two. When $r \geq 1$, we use
Table 1 Summary of proposed lognormal based slew and delay metrics for step and ramp inputs. All formulae are explicit functions of the whole swing input slew \( T \) and the first three circuit moments.

For delay, PERI uses a parameter \( \alpha \) to reflect the degree of significance of the ramp on the step metric. As the input slew approaches infinity, the delay asymptotically approaches the Elmore delay. Hence, the value of \( \alpha \) is chosen such that \( \alpha = 1 \) when the full swing input slew \( T \) is zero (thereby yielding the step delay metric) and \( \alpha \rightarrow 0 \) when \( T \rightarrow \infty \) (yielding the Elmore delay). The lognormal formula \( \text{LnD}(T) \) is given by

\[
\text{LnD}(T) = \alpha \text{LnD} - m_1 (1 - \alpha)
\]

where

\[
\alpha = \frac{\left(2m_2 - m_1^2\right)}{\left(2m_2 - m_1^2 + T^2/12\right)}^{5/2}
\]

Note that the formula for \( \alpha \) achieves the desired asymptotic behavior. The formula derives from the assumption that the Pearson Skewness Coefficient for the PDF for step response matches that of the ramp response.

For slew metrics, PERI proposes a root-mean square relationship between the step slew. This gives a slew formula of

\[
\text{LnS}(T) = \sqrt{\text{LnS}^2 + (0.8)^2 T^2}.
\]

Table 1 consolidates all the formulae into a single table in which each is given as an explicit function of the first three circuit moments and the whole swing input slew \( T \).

6. EXPERIMENTAL RESULTS

To verify the effectiveness of the lognormal-based delay and slew metrics, we extracted 432 routed nets containing 2244 sinks from an industrial ASIC part in 0.18 micron technology. The nets were chosen by a filtering process that required the maximum sink delay to be at least 10 ps and for the ratio of the closest sink to the furthest sink in the net to be less than 0.25. This ensures that each net has at least one “near-end” sink. We classify the 2244 total sinks into three categories:

- 1187 far-end sinks have delay greater or equal to 75% of the maximum delay to the furthest sink in the net,
- 670 mid-end sinks which have delay between 25% and 75% of the maximum delay, and
- 367 near-end sinks which have delay less than or equal to 25% of the maximum delay.

For each sink we compute delay and slew according to SPICE and measure the relative error of the appropriate metric to the SPICE result. We average the absolute values over all the errors and report the average relative error over all sinks in the following tables. We hook up a driver to the source of each RC network, where the driver is a voltage source (excited with a step response) followed by a resistor.

### 6.1 Experiments for Delay

LnD is a constant factor of two percent larger than D2M, and D2M was shown to be quite effective compared to previous delay metrics in [1]. However, those experiments were performed on randomly generated networks. To demonstrate the effectiveness for nets from a real design, we revisit the comparisons to the best randomly generated networks. To demonstrate the effectiveness for nets from a real design, we revisit the comparisons to the best randomly generated networks.
• The 2CM metric is actually quite accurate at the far end (averaging roughly 1.5% error over all inputs with very low standard deviation).

- At the far-end, LnD is quite accurate, averaging roughly 1.5% error over all inputs with very low standard deviation.
- 1NR is the most accurate at the near-end but is inadequate for the far-end, frequently yielding negative delays.
- The KM metric, while more accurate than Elmore, is not as accurate as LnD, especially at the near-end.

Overall, LnD has by far the smallest relative error and standard deviation of the four metrics.

6.2 Experiments for Slew

There are only two close form slew metrics in the literature. For 10/90 slew, Bakoglu [3] proposed using \( \ln(9) \) times the RC delay (we use Elmore). We call this first order approximation Bak. The other close form metric, as noted by [4] and [5] is simply \( 385 \pi / 9 \) times a constant. Elmore suggested a constant of \( 1/2\sigma \), which is not particularly accurate. A better choice is a constant of \( \ln(9) \) since it is exact for a single RC network and the constant that results when matching the variance of the impulse response to an exponential distribution. We call this metric 2CM for the second central moment. Finally, although not a closed-form metric, we also compare to WED since it enables sampling at the 10% and 90% delay points. The results are shown in Table 3. We observe the following:

<table>
<thead>
<tr>
<th>Driver Resistance = 0 Ohms</th>
<th>Average % Relative Error</th>
<th>Standard Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sinks LnS Bak 2CM WED LnS Bak 2CM WED</td>
<td></td>
<td></td>
</tr>
<tr>
<td>near 98.88 141.6 1734 421.1 61.98 95.38 1362 334.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mid 5.38 19.04 38.64 9.86 5.18 7.77 33.71 5.44</td>
<td></td>
<td></td>
</tr>
<tr>
<td>far 0.51 12.50 2.75 3.24 0.63 6.02 3.70 2.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total 18.74 36.43 319.1 79.39 46.00 64.16 881.6 214.5</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Driver Resistance = 100 Ohms</th>
<th>Average % Relative Error</th>
<th>Standard Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sinks LnS Bak 2CM WED LnS Bak 2CM WED</td>
<td></td>
<td></td>
</tr>
<tr>
<td>near 35.66 29.80 304.3 56.20 32.02 32.13 200.9 59.89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mid 5.92 20.32 51.94 11.47 4.96 8.30 29.28 6.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>far 0.49 12.67 2.70 3.17 0.53 6.02 3.52 2.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total 5.68 15.72 47.47 11.00 16.60 14.33 124.8 28.15</td>
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</table>

<table>
<thead>
<tr>
<th>Driver Resistance = 200 Ohms</th>
<th>Average % Relative Error</th>
<th>Standard Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sinks LnS Bak 2CM WED LnS Bak 2CM WED</td>
<td></td>
<td></td>
</tr>
<tr>
<td>near 10.89 22.45 175.0 16.69 36.97 39.09 221.5 68.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mid 6.42 24.09 59.75 12.10 3.13 6.61 28.96 5.13</td>
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<td></td>
</tr>
<tr>
<td>far 0.45 12.43 2.44 3.04 0.40 6.01 3.08 1.93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total 1.85 14.97 17.14 5.12 5.75 9.29 44.85 10.23</td>
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</table>

Table 3 Slew comparisons for the lognormal metric.

We observe the following:

• The 2CM metric is actually quite accurate at the far end (averaging 2-3% relative error). However, it is completely inadequate at the near-end, grossly overestimating slew.

- LnD tends to overestimate delay, though LnD’s relative error is much smaller than Elmore. While Elmore delays are on average four to six times more than SPICE at the near-end, LnD overestimates close to a factor of two.
- WED is also fairly accurate at both the mid- and far-end, but very inaccurate at the near-end (especially for drivers with low resistance) though not to the degree of 2CM. Indeed, since WED samples the actual 10% and 90% delay points, its accuracy is highly sensitive to where the 90% point lies on the tail of the highly-skewed impulse response curve.
- For a first order approximation, the Bakoglu metric is actually quite nice, having accuracy similar to that of Elmore for delay. It has better accuracy than both 2CM and WED at the near-end, though is quite a bit worse at the far-end.
- Finally, the LnS metric clearly dominates the other three metrics in terms of relative error and standard-deviation. LnS is on average within 1% of the optimal solution at the far-end and is also within a factor of two of optimal at the near-end where slew measurements are highly sensitive.

7. Conclusions

We have used the lognormal distribution to derive closed form formulae for both delay and slew that can be used with either a step or ramp input. We have made it easy for anyone to implement by consolidating all the formulae in Table 1. Our experiments demonstrate significant accuracy advantages over previous approaches. In future work, we seek to implement these metrics within a static timing analyzer and a physical synthesis engine.

REFERENCES