MODEL REDUCTION FOR DC SOLUTION OF LARGE NONLINEAR CIRCUITS

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ABSTRACT

A new algorithm based on model reduction using Krylov subspace technique is proposed to compute the DC solution of large nonlinear circuits. The proposed method combines continuation methods with model reduction techniques. Thus it enables the application of the continuation methods to an equivalent reduced-order set of nonlinear equations instead of the original system. This results in a significant reduction in the computational expense as the size of the reduced equations is much less than that of the original system.

The reduced order system is obtained by projecting the set of nonlinear equations, whose solution represents the DC operating point, into a subspace of a much lower dimension. It is also shown that the both the reduced-order system and the original system share the first $q$ derivatives w.r.t. the circuit variable used to parameterize the family of the solution trajectories generated by the continuation method.

1. INTRODUCTION

The tendency towards ever larger circuits and more complex devices is stretching the limits of the current CAD techniques. The DC solution poses a particular problem as it requires the solution of a large set of nonlinear equations through some iterative techniques such as Newton’s method [1]. However, the convergence of Newton’s method requires an initial starting point that is sufficiently close to the solution. Continuation methods, also known as homotopy methods, have been introduced to relieve the user of having to guess a good starting point [2]-[4]. The basic idea behind the continuation methods involves replacing the original system of nonlinear equations with another system of the same size but whose solution is trivial, finding the operating point for the easier system and then incrementally sweeping some circuit parameter to generate a trajectory of solutions. The terminus of the sweep is the operating point of the original circuit. Although these methods remove the burden of having to guess a good starting point for robust convergence, they are computationally expensive as they require the solution of a large set of nonlinear equations at each incremental value of the parameter being swept.

On the other hand, the introduction of Krylov-based model reduction techniques for linear systems has had a tremendous impact on both accuracy and efficiency [5]-[7]. Only recently, Krylov subspace methods have been extended to the transient simulation of large nonlinear circuits [8].

In this paper, we present a new algorithm that combines model reduction techniques based on Krylov subspace methods with continuation methods for obtaining the DC operating point of a circuit. The algorithm presented in this paper is based on projecting the full set of nonlinear equations, whose solution represents the DC operating point, into a Krylov subspace of lower dimension. As a result, the original set of nonlinear equations is replaced by a much smaller set of nonlinear equations such that the first $q$ derivatives w.r.t. the sweeping parameter of the original system are preserved by the small system. The solution of the reduced system is then traced as the sweeping parameter is varied. This results in a significant reduction in the CPU time due to the fact that a much smaller system need to be solved at each incremental value of the sweeping parameter.

The paper is organized as follows. Section 2 describes formulating the Krylov subspace and the derivation of the reduced-order nonlinear model. Section 3 describes the basic outlines for the proof that the proposed model-reduction algorithm preserves the first $q$ derivatives w.r.t. the parameter being swept. Sections 4 and 5 present numerical results and conclusion, respectively.

2. REDUCTION OF THE NONLINEAR EQUATIONS

The main idea of the new algorithm is centered around the introduction of the model reduction techniques to the solution of a large set of nonlinear equations through continuation methods. This will enable us to use some version of the continuation methods (e.g. the source stepping) to a reduced system of nonlinear equations instead of the original system. The reduced system shares the same first $q$ derivatives w.r.t. the sweeping parameter.

The main steps involved in the new algorithm are as follows: (1) Formulation of the modified nodal analysis (MNA) circuit equations including the nonlinear elements, (2) Computation of the derivatives w.r.t. the sweeping parameter, (3) Formulation of the Krylov subspace using the derivatives computed in step (2) and obtaining the modified congruent transformation matrix ($Q$) using orthonormalization and (4) Model reduction on the entire network using congruent transformation. Details of the algorithm are discussed briefly in the following sections.
2.1 Continuation Through Source Stepping

The DC operating point problem for a nonlinear network $\Phi$ can be written using MNA formulation as [9]:

$$\Phi(x) \equiv Gx + F(x) - b = 0 \quad (1)$$

where,

- $x \in \mathbb{R}^{N_v}$ is the vector of node voltages appended by independent voltage sources currents,
- $G \in \mathbb{R}^{N_v \times N_v}$ is a constant matrix describing lumped memoryless elements of the network,
- $b \in \mathbb{R}^{N_v}$ is a vector with entries determined by the independent voltage/current sources,
- $F(x) \in \mathbb{R}^{N_v}$ is a function describing the nonlinear elements of the circuit, and
- $N_\Phi$ is the number of variables in the MNA formulation.

The basic idea of the continuation methods is to augment (1) with some circuit parameter, say $\alpha$, $H(x, \alpha)$, where $\alpha$ is chosen so that the solution of $H(x, 0)$ is easy to obtain and $H(x, 1) = \Phi(x)$ identically in $x$. In the source stepping scheme $\alpha$ is chosen to be a multiplier of the source vector:

$$H(x, \alpha) \equiv Gx + F(x) - \alpha b = 0 \quad (2)$$

At $\alpha = 0$ the circuit has an obvious operating point at which each node voltage is zero. Starting from this point, small increments in $\alpha$ are taken, at each point finding an appropriate $x$ to satisfy $H(x, \alpha) = 0$, until $\alpha = 1$. In the next subsection we describe how to obtain the derivatives of $x$ w.r.t. $\alpha$.

However, it is to be noted that other approaches for the continuation methods might formulate the DC problem differently. For example, in [4] the following formulation has been considered:

$$H(x, \alpha) \equiv \alpha \Phi(x) + (1 - \alpha)(x - a) = 0 \quad (3)$$

where $a \in \mathbb{R}^{N_v}$ is some constant vector. In this paper we will be concerned only with the formulation in (2).

2.2 Computation of the derivatives w.r.t. $\alpha$

$x$ in (1) is an implicit function of $\alpha$ and may be expanded in a Taylor series as:

$$x(\alpha) = \sum_{k=0}^{\infty} a_k (\alpha - \alpha_0)^k \quad (4)$$

where $a_k = x^{(k)}/k!$, $k = 0, 1, 2, \ldots$ are the normalized derivatives w.r.t. $\alpha$ at $\alpha = \alpha_0$ and are computed using

$$Ga_k + f_k - b = 0 \quad (5)$$

where $f_i$ denotes the $i^{th}$ normalized derivative of $F(x)$ w.r.t. $\alpha$ evaluated at $\alpha = \alpha_0$ and $a_0$ is assigned the value of $x$ at $\alpha = \alpha_0$. It is clear that the coefficients $a_k, k = 1, 2, \ldots$ can be computed recursively using (5) and (6). The first derivative of $F(x)$ w.r.t. $\alpha$ is computed as:

$$\frac{d}{d\alpha}F(x(\alpha)) = J\left(\sum_{k=0}^{\infty} a_k \alpha^k\right) \sum_{k=1}^{\infty} ka_k \alpha^{k-1} \quad (7)$$

where $J(x) = \frac{d}{dx}F(x)$. Using (8) and for simplicity assigning $\alpha_0 = 0$, the $m^{th}$ derivative of $F(x)$ at $\alpha = 0$ can be computed as

$$\frac{d^m}{d\alpha^m}F(x(\alpha)) = (m!)f_m = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-j-1)!} [\zeta^{m-j-1}(0)] a_{j+1} \quad (8)$$

As an example of the calculation of the derivatives, consider

$$F(v_d) = I_s (\exp(v_d/V_T) - 1) \quad (9)$$

where $v_d = \sum_j c_j i^j$ and $F(v_d) = \sum_i d_i \alpha^i$. Expansion of the exponential gives

$$d_\alpha = I_s (\exp(c_0/V_T) - 1) \quad (10)$$

$$d_i = \frac{1}{i!V_T} \sum_{j=0}^{i-1} d_j c_{i-j} (i-j) + \frac{1}{V_T} c_i \quad (11)$$

Similar expansions can be found for other functions [10]. Using (7) and (8), (5) and (6) become

$$(G + \zeta(0)) a_1 = b \quad (12)$$

$$(G + \zeta(0)) a_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k^2} \zeta^{k-j-1}(0) a_{j+1} = 0 \quad (k > 1) \quad (13)$$

Thus to obtain the derivatives we need to perform one LU decomposition of the matrix $G + \zeta(0)$ and then each derivative can be obtained efficiently by one forward and one backward substitution. Thus the complexity of computing $k$ derivatives is $O(kN_\Phi)$.

2.3 Reduced Model via Congruent Transformation

The original system (2) is reduced to a smaller set of unknowns through a congruent transformation obtained from the Krylov subspace $K$. This subspace, formed by the derivatives computed in (13) and (14), is defined as:

* Other sophisticated implementation schemes will parametrize the solution curve by the arc length parameter [4].
\[ K = [a_0 a_1 \ldots a_q] \] (14)

where \( q \) is the order of the reduced system \( q \ll N \). Performing an orthogonal decomposition [5]-[8] on \( K \) we have

\[ K = QR \] (15)

where \( Q^TQ = U_q \) and \( U_q \in \mathbb{R}^{q\times q} \) is an identity matrix. Using the matrix \( Q \) obtained from (15) we perform a congruent transformation on the original system (2) as

\[ x = Q\hat{x} \] (16)

where \( \hat{x} \in \mathbb{R}^q \). This change of variables \( (x \rightarrow \hat{x}) \) reduces the original system (2) to a system of a smaller set of unknowns, given by

\[ \hat{H}(x, \alpha) = \hat{G}\hat{x} + \hat{F}(\hat{x}) - \alpha\hat{b} = 0 \] (17)

where,

\[ \hat{G} = Q^TQ; \quad \hat{F}(\hat{x}) = Q^TF(Q\hat{x}); \quad \hat{b} = Q^Tb \] (18)

The solution of the reduced set of equations (17) can then be tracked as \( \alpha \) is varied from 0 to 1. The solution of the original system (2) is obtained using the transformation given in (16). It is to be noted that the computational cost in solving (17) for incremental changes in \( \alpha \) is drastically reduced when compared to the solution of (2) as the order of the reduced system is significantly less than the order of the original system.

The error due to the solution obtained from the small system can be computed as the \( l2 \) norm of the current error vector

\[ \varepsilon = ||H(x, \alpha)|| \] (19)

If this error grows beyond a certain prescribed value, say \( \varepsilon_1 \), then a correction step using Newton-Raphson iterations for the original system may be needed using that current solution as a seed.

3. PROOF OF EQUVALENCE OF DERIVATIVES

Proof that the reduced order system (17) preserves the first \( q \) derivatives of the large system (2) w.r.t. \( \alpha \) is given by Mathematical Induction. First it shall be proved that the first derivative obtained from the reduced system is equivalent to that obtained from the large system. Next, we will show that the \( k^{th} \) derivative is conserved if the previous \( (k-1) \) derivatives are conserved. \( \hat{x}(\alpha) \) in (17) is expanded in a Taylor series as

\[ \hat{x}(\alpha) = \sum_{k=0}^{\infty} \hat{a}_k\alpha^k \] (20)

The coefficients \( \hat{a}_k \) are computed using the recursive relationship

\[ \hat{G}\hat{a}_1 + \hat{f}_1 - \hat{b} = 0 \] (21)

\[ \hat{G}\hat{a}_k + \hat{f}_k = 0 \quad (k > 1) \] (22)

Through mathematical manipulation, it can be shown that:

\[ \hat{a}_1 = a_1 \] (23)

which shows that the first derivative is conserved. In a similar manner we can proceed to show that if the hypothesis holds well for \( l=1 \) to \( l=k-1 \), i.e. \( \hat{a}_l = a_l \) for \( 1 \leq l \leq k-1 \), then it also holds well for \( l=k \), i.e. \( \hat{a}_k = a_k \).

4. NUMERICAL RESULTS

Example 1

In this example we demonstrate the equivalence between the derivatives w.r.t. \( \alpha \) as obtained from the reduced-order system and that of the large original system. For this purpose, a network of exponentially behaved nonlinear resistors was considered for this example. The size of the MNA Matrix of the original circuit is 100x100. The proposed algorithm was used to construct a reduced order system by considering the first ten derivatives to form the Krylov subspace. The new model was then used to compute the derivatives of the original system. The derivatives thus computed were compared with that of the original system.

Table 1: Comparison of normalized derivatives

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<th>Original System</th>
<th>Reduced-Order System</th>
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Example 2

A ladder circuit consisting of 150 diodes and linear resistors has been considered for this example. The original size of the original MNA matrix has been (151x151). The DC operating point for this circuit was obtained after 665 iteration using conventional Newton-Raphson iterations to solve the original system. Using the proposed algorithm, on the other hand, the
original system was reduced to a system of size (7x7). The reduced system thus generated was solved for values of $\alpha = 0 \rightarrow 1$ using the source stepping technique. The reduced system required correction at three intermediate points ($\alpha$=0.01, 0.02 and 0.22). The number of Newton-Raphson iterations of the original system that were required at the correction points was only 13 iterations.

Example 3

In this example a circuit consisting of 30 cascaded OpAmp (CA3096) stages has been considered. The total number of transistors is 150 which results in MNA matrix of size (664x664). Fig. 1 shows a schematic for the CA3096 OpAmp. The nonlinear system of equations did not converge using the conventional Newton-Raphson iterations. Using the algorithm presented above, the circuit was reduced to a system of size (10x10). The reduced-order system thus generated has been solved for values of $\alpha = 0 \rightarrow 1$ using the source stepping technique. The reduced-order system required correction at four intermediate points ($\alpha$=0.04, 0.05, 0.08 and 0.24) which required only 13 Newton-Raphson iterations on the original system. Table 2: outlines the main computational cost as compared between the conventional Newton-Raphson method and the new model reduction algorithm.

5. CONCLUSION

In this paper a novel algorithm has been presented for obtaining the DC operating point through Krylov subspace Model reduction techniques. The reduced-order system of nonlinear equations can be solved using any of the continuation methods. Significant reduction of the computational cost can be achieved as the order of the small system is much less than the original system.

REFERENCES


<table>
<thead>
<tr>
<th>Examples</th>
<th>Size of the original MNA matrix</th>
<th>Size of the reduced-order system</th>
<th># of LU decomp. of the original MNA matrix</th>
<th>Conventional N-R</th>
<th>The new algorithm</th>
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<td>(10x10)</td>
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Table 2: Main Computational Cost.

Fig. 1 Schematic layout of the CA3096 OpAmp.