Model-Reduction of Nonlinear Circuits Using Krylov-Space Techniques

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1. ABSTRACT

A new algorithm based on Krylov subspace methods is proposed for model-reduction of large nonlinear circuits. Reduction is obtained by projecting the original system described by nonlinear differential equations into a subspace of a lower dimension. The reduced model can be simulated using conventional numerical integration techniques. Significant reduction in computational expense is achieved as the size of the reduced equations is much less than that of the original system.

1.1 Keywords

Model-reduction, nonlinear circuits, Krylov-subspace

2. INTRODUCTION

The tendency towards ever larger circuits and more complex devices is stretching the limits of current simulation methods. Large *nonlinear* circuits pose a particular problem for the time domain simulation as most commonly used simulators are based on implicit numerical integration methods which require solution of large set of nonlinear algebraic equations at each time point. Hence, it is clear that some way must be found to increase computational efficiency without sacrificing the analysis accuracy. Several attempts to achieve this goal have been reported previously in the literature [1]-[8].

On the other hand, in the case of *linear* circuits, several efficient model-reduction techniques have been reported recently [9]-[13]. Most of these techniques have been based on projecting the linear system into a subspace of lower dimension. Significant reduction in computational complexity was obtained using Krylov subspace methods. It was also shown that projection into a Krylov subspace is equivalent to matching the frequency domain moments.

In this paper, we present a method based on Krylov subspace to obtain reduced-order models for large nonlinear circuits. An algorithm is described for projecting the full *nonlinear* circuit equations into a Krylov subspace of lower dimension. As a result, the original set of nonlinear time-domain equations is replaced by a smaller set of nonlinear equations such that the first ‘*q*’ derivatives of the time-response of the original system and the response obtained using the reduced system are identical. The reduced system can be solved using any of the conventional numerical integration techniques, resulting in significant reduction in the CPU time due to the fact that at each time point, a much smaller set of nonlinear algebraic equations need to be solved.

This paper is organized as follows. Section 2 presents a systematic approach to form the Krylov subspace and derive the reduced-order nonlinear model. Section 3 presents a proof that the proposed model-reduction algorithm preserves the first ‘*q*’ derivatives of the time-domain response. Sections 4 and 5 present numerical results and conclusions, respectively.

3. MODEL-REDUCTION OF NONLINEAR CIRCUITS

The main idea of the new algorithm presented in this paper is centered around the replacement of the original nonlinear circuit equations by a smaller set of nonlinear equations such that the first ‘*q*’ derivatives of the time-response of the original system and the response obtained from the reduced system are identical. Details of the algorithm are discussed briefly in the following sections.

3.1 Computation of Time Domain Derivatives of the Original System

Consider a general network ϕ containing an arbitrary number of linear and nonlinear components. Without loss of generality, the MNA formulation for the network ϕ can be written as [15]

$$\boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{G}\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{t}) \quad \textbf{(1)}$$

where, $x \in \Re^{N_\phi}$ is the vector of node voltage waveforms appended by independent voltage source currents, linear inductor currents, nonlinear capacitor charges, nonlinear inductor flux waveforms and currents and voltages due to nonlinear components, $\boldsymbol{C} \in \Re^{N_\phi \times N_\phi}$ and $\boldsymbol{G} \in \Re^{N_\phi \times N_\phi}$ are constant matrices describing the lumped memory and memoryless elements of the network, respectively. $\boldsymbol{b}(\boldsymbol{t}) \in \Re^{N_\phi}$ is a vector with entries determined by the independent voltage/current sources, $\boldsymbol{F}(\boldsymbol{x}) \in \Re^{N_\phi}$ is a function describing the nonlinear elements of the circuit and N_ϕ is the total number of variables in the MNA formulation.

x(t) in (1) is expanded in a Taylor series as

$$\boldsymbol{x}(t) = \sum_{k=0} \boldsymbol{a}_k (t - t_0)^k \quad \textbf{(2)}$$

where $\boldsymbol{a}_k = \boldsymbol{x}^{(k)}(t_0)/k!, k = 0, 1, 2, \ldots$ are the normalized time domain derivatives of $\boldsymbol{x}$ and are computed using

$$(k+1)\boldsymbol{C}\boldsymbol{a}_{k+1} + \boldsymbol{G}\boldsymbol{a}_{\boldsymbol{k}} + \boldsymbol{f}_{\boldsymbol{k}} = \boldsymbol{b}_{\boldsymbol{k}} \quad \textbf{(3)}$$

where $\boldsymbol{b}_i$ and $\boldsymbol{f}_i$ denote the i^{th} derivatives of $\boldsymbol{b(t)}$ and $\boldsymbol{F(x)}$ evaluated at $t = t_0$ respectively and $\boldsymbol{a}_0$ is assigned the initial conditions $\boldsymbol{x}_0$. Assuming for the moment that the matrix $\boldsymbol{C}$ is invertible, the coefficients $\boldsymbol{a}_k, k = 1, 2, \ldots$ can be computed recursively using (3). The first derivative of $\boldsymbol{F(x)}$ is computed as

$$\frac{d}{dt}\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{J}\left(\sum_{k=0} \boldsymbol{a}_k t^k\right)\sum_{k=1} k\boldsymbol{a}_k t^{k-1} = \zeta(t)\sum_{k=1} k\boldsymbol{a}_k t^{k-1} \quad \textbf{(4)}$$

where $\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$.

Using (4) and for simplicity assigning $t_0 = 0$, the m^{th} derivative of $\boldsymbol{F(x)}$ at $t = 0$ can be computed as [16]

$$\boldsymbol{f}_m = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-j-1)!}(j+1)[\zeta^{m-j-1}(0)]\boldsymbol{a}_{j+1} \quad \textbf{(5)}$$

Using (5), (3) becomes

$$(k+1)\boldsymbol{C}\boldsymbol{a}_{k+1} + \boldsymbol{G}\boldsymbol{a}_{\boldsymbol{k}} + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k}[\zeta^{k-j-1}(0)]\boldsymbol{a}_{j+1} = \boldsymbol{b}_{\boldsymbol{k}}; \qquad k \geq 1 \quad \textbf{(6)}$$

In the case where $\boldsymbol{C}$ is not invertible, (3) may be separated into

$$(k+1)\begin{bmatrix} \boldsymbol{C}_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_{k+1} \\ \boldsymbol{w}_{k+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{w}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}_u(u_k, w_k) \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_{u,k} \\ \boldsymbol{b}_{w,k} \end{bmatrix} \quad \textbf{(7)}$$

where $\boldsymbol{a}_k$ has been divided into the two disjoint differential and algebraic sets $\boldsymbol{u}_k$ and $\boldsymbol{w}_k$ such that there are no empty rows or columns in $\boldsymbol{C}_{11}$. The first row of (7) is an explicit equation for $\boldsymbol{u}_{k+1}$ in terms of $\boldsymbol{u}_k$ and $\boldsymbol{w}_k$ requiring $\boldsymbol{C}_{11}$ to be invertible. The second row of (7) is an explicit equation for $\boldsymbol{w}_k$ in terms of $\boldsymbol{u}_k$ and $\boldsymbol{b}_k$ requiring $\boldsymbol{G}_{22}$ to be invertible. The required invertible conditions are usually met by the modified nodal analysis formulation used to develop circuit equations. Special formulation methods can be used to ensure that $\boldsymbol{C}_{11}$ and $\boldsymbol{G}_{22}$ are invertible without the need for state space formulation [17].

3.2 Reduced Model via Congruent Transformation

The original system (1) is reduced to a smaller set of unknowns through a congruent transformation obtained from the Krylov subspace $\boldsymbol{K}$. This subspace, formed by the derivatives computed in (6) is defined as

$$\boldsymbol{K} = [\boldsymbol{a}_0 \ \ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_q] \quad \textbf{(8)}$$

where q is the order of reduction and $q \ll N_\phi$. Performing an orthogonal decomposition [14] on $\boldsymbol{K}$ we have

$$\boldsymbol{K} = \boldsymbol{Q}\boldsymbol{R} \quad \textbf{(9)}$$

where $\boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{Q} = \boldsymbol{U}_q$ and $\boldsymbol{U}_q \in \Re^{qxq}$ is an identity matrix. Using the matrix $\boldsymbol{Q}$ obtained from (9) we perform a congruent transformation on the original system (1) given by

$$\boldsymbol{x} = \boldsymbol{Q}\hat{\boldsymbol{x}} \quad \textbf{(10)}$$

where $\hat{\boldsymbol{x}} \in \Re^q$. This change of variable $(\boldsymbol{x} \to \hat{\boldsymbol{x}})$ reduces the original system (1) to a system with a smaller set of unknowns, given by

$$\hat{\boldsymbol{C}}\dot{\hat{\boldsymbol{x}}} + \hat{\boldsymbol{G}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{b}}(\boldsymbol{t}) \quad \textbf{(11)}$$

where

$$\hat{\boldsymbol{G}} = \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{G}\boldsymbol{Q}; \qquad \hat{\boldsymbol{C}} = \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{C}\boldsymbol{Q};$$
$$\hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}) = \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{F}(\boldsymbol{Q}\hat{\boldsymbol{x}}); \qquad \hat{\boldsymbol{b}}(t) = \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{b}(\boldsymbol{t}) \quad \textbf{(12)}$$

The reduced set of equations (11) can be solved using any of the conventional numerical integration techniques [17] to get $\hat{\boldsymbol{x}}$. The solution for the original system (1) is obtained using the transformation given in (10). It is to be noted that the computational cost involved in solving (11) is drastically reduced when compared to (1) as the order of the reduced model is significantly less than the order of the original system.

4. PROOF OF PRESERVATION OF DERIVATIVES

Proof that the reduced-order system (11) preserves the first q derivatives of the original system (1) is given by *Mathematical Induction*. First it shall be proved that the first derivative obtained from the reduced-order model is equivalent to that obtained from the original system (1). Next, we will show that the $(k+1)^{th}$ derivative is conserved if the previous k derivatives are conserved.

$\hat{\boldsymbol{x}}(t)$ in (11) is expanded in a Taylor series as

$$\hat{\boldsymbol{x}} = \sum_{\boldsymbol{k}=\boldsymbol{0}} \hat{\boldsymbol{a}}_{\boldsymbol{k}}\boldsymbol{t}^{\boldsymbol{k}} \quad \textbf{(13)}$$

The coefficients $\hat{\boldsymbol{a}}_{\boldsymbol{k}}$ are computed using the recursive relationship

$$(k+1)\hat{\boldsymbol{C}}\hat{\boldsymbol{a}}_{k+1} + \hat{\boldsymbol{G}}\hat{\boldsymbol{a}}_{\boldsymbol{k}} + \hat{\boldsymbol{f}}_{\boldsymbol{k}} = \hat{\boldsymbol{b}}_{\boldsymbol{k}} \quad \textbf{(14)}$$

where $\hat{\boldsymbol{a}}_0$ represents the initial conditions of the reduced system and is chosen such that it satisfies

$$\boldsymbol{Q}\hat{\boldsymbol{a}}_0 = \boldsymbol{a}_0 \quad \textbf{(15)}$$

From (14), it is seen that $\hat{\boldsymbol{a}}_1$ is given by

$$\hat{\boldsymbol{C}}\hat{\boldsymbol{a}}_1 + \hat{\boldsymbol{G}}\hat{\boldsymbol{a}}_0 + \hat{\boldsymbol{F}}(\hat{\boldsymbol{a}}_0) = \hat{\boldsymbol{b}}_0 \quad \textbf{(16)}$$

i.e.,

$$Q^T C Q \hat{a}_1 + Q^T G Q \hat{a}_0 + Q^T F(Q\hat{a}_0) = Q^T b_0 \tag{17}$$

Using (9) and (15), (17) can be written as

$$K^T (C[a_0 \;\; a_1 \;\; \dots \;\; a_q] R^{-1} \hat{a}_1 + G a_0 + F(a_0) - b_0) = 0 \tag{18}$$

Substituting $R^{-1}\hat{a}_1 = e_2$, where e_r is the r^{th} column of the identity matrix $U_q \in \Re^{qxq}$ and simplifying (18) using (3), we have

$$K^T (C a_1 + G a_0 + F(a_0) - b_0) = 0 \tag{19}$$

Hence, $R^{-1}\hat{a}_1 = e_2$ is a solution of (18). Therefore the first derivative of the original system obtained from the reduced system is

$$Q\hat{a}_1 = K R^{-1} \hat{a}_1 = [a_0 \;\; a_1 \;\; \dots \;\; a_q] e_2 = a_1 \tag{20}$$

From (20) it can be seen that the first derivative of the system is conserved.

We now proceed to prove that if the hypothesis holds good for all l from $l=0$ to $l=k$, i.e. $Q\hat{a}_l = a_l$ for $0 \le l \le k$, then it also holds good for $l = k+1$, i.e. $Q\hat{a}_{k+1} = a_{k+1}$.

The first derivative of $\hat{F}(\hat{x})$ is computed as

$$\frac{d}{dt}\hat{F}(\hat{x}) = Q^T \frac{\partial}{\partial x}(F(x)) Q \frac{\partial}{\partial t}\hat{x} = Q^T \zeta(t) Q \sum_{k=1} k \hat{a}_k t^{k-1} \tag{21}$$

Using (21), the m^{th} derivative of $\hat{F}(\hat{x})$ at $t = 0$ is computed to be

$$\hat{f}_m = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-j-1)!}(j+1) Q^T [\zeta^{m-j-1}(0)] Q \hat{a}_{j+1} \tag{22}$$

Using (22), (14) becomes

$$(k+1)\hat{C}\hat{a}_{k+1} + \hat{G}\hat{a}_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k} Q^T [\zeta^{k-j-1}(0)] Q \hat{a}_{j+1} = \hat{b}_k; \;\; k \ge 1 \tag{23}$$

Using (9) and (12), (23) can be written as

$$Q^T \left((k+1) C [a_0 \;\; a_1 \;\; \dots \;\; a_q] R^{-1} \hat{a}_{k+1} + G Q \hat{a}_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k} [\zeta^{k-j-1}(0)] Q \hat{a}_{j+1} - b_k \right) = 0 \tag{24}$$

Since $Q\hat{a}_l = a_l$ for $0 \le l \le k$, (24) is reduced to

$$K^T \left((k+1) C [a_0 \;\; a_1 \;\; \dots \;\; a_q] R^{-1} \hat{a}_{k+1} + G a_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k} [\zeta^{k-j-1}(0)] a_{j+1} - b_k \right) = 0 \tag{25}$$

Substituting $R^{-1}\hat{a}_{k+1} = e_{k+2}$ in (25), and simplifying using (6) we have

$$K^T \left((k+1) C a_{k+1} + G a_k + \sum_{j=0}^{k-1} \frac{(j+1)}{(k-j-1)!k} [\zeta^{k-j-1}(0)] a_{j+1} - b_k \right) = 0 \tag{26}$$

Hence $R^{-1}\hat{a}_{k+1} = e_{k+2}$ is a solution of (25). Therefore the *(k+1)*th derivative of the original system obtained from the reduced system is

$$Q\hat{a}_{k+1} = K R^{-1} \hat{a}_{k+1} = [a_0 \;\; a_1 \;\; \dots \;\; a_q] e_{k+2} = a_{k+1} \tag{27}$$

From (27) it can be seen that the *(k+1)*th derivative of the system is conserved if all it's previous derivatives are conserved. Thus by *Mathematical Induction* we can conclude that the first q derivatives are conserved for the system.

5. NUMERICAL RESULTS

5.1 Example 1

A lumped nonlinear circuit with all it's capacitors initially charged to a unit voltage was considered for this example. The size of the MNA matrix was 100 x 100. The proposed algorithm was used to find the reduced-order model of the system by considering the derivatives of the nonlinear network to form the Krylov subspace. The response of the network was computed using the reduced-order model. As a sample of the results, comparison with the output across one of the nonlinear components, obtained from the original system is shown in Figures 1 and 2. As expected, it is observed that the accuracy of the response increases as we increase the number of derivatives considered in the formulation of the Krylov subspace.

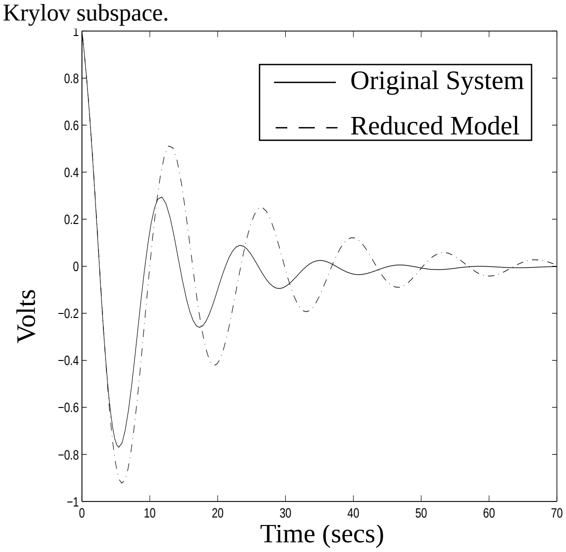

Figure 1. Response of the Reduced Model for q=13

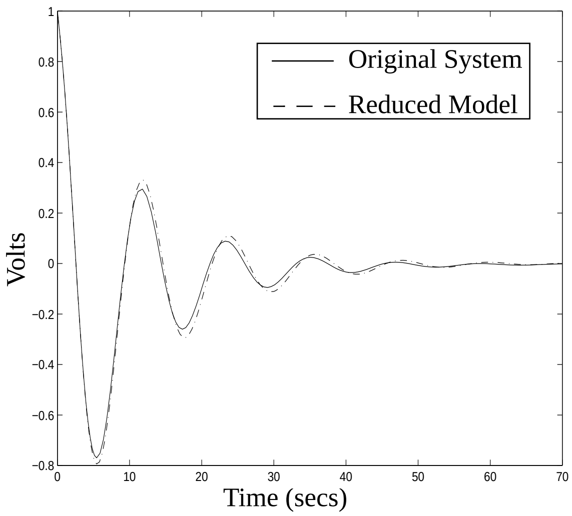

Figure 2. Response of the Reduced Model for q=15

5.2 Example 2

Another lumped network consisting of exponentially behaved nonlinear resistors was considered for this example. The size of the MNA matrix of the original circuit was 1000 x 1000. The proposed algorithm was used to find the reduced-order model of the system by considering 25 derivatives of the nonlinear network to form the Krylov subspace. It was observed that using the proposed algorithm, the size of the reduced model was approximately 2.5% of the size of the original circuit. The response of the original circuit was computed using the reduced-order model. As a sample of the results, comparison with an output across one of the nodes, obtained from the original system is shown in Figure 3 and as seen both the responses match accurately.

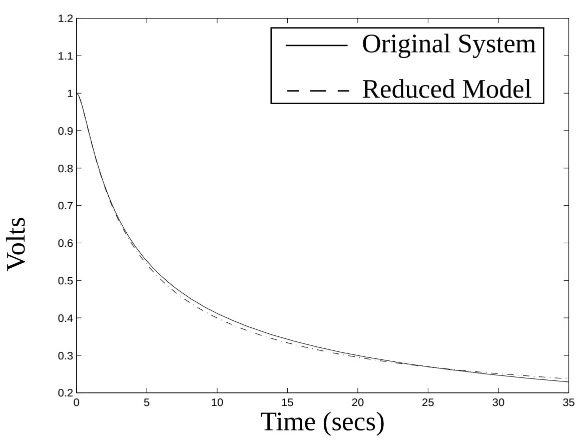

Figure 3. Transient response comparison

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