ON CONVEX FORMULATION OF THE FLOORPLAN AREA MINIMIZATION PROBLEM

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ABSTRACT

It is shown that the floorplan area minimization problem can be formulated as a convex programming problem with the numbers of variables and constraints significantly less than those previously published.

1. INTRODUCTION

Floorplanning is a key step in the VLSI physical design cycle. Its main task is to place blocks with fixed area but unknown dimensions on a chip in such a way that the area of the enveloping rectangle is minimized. Usually, the blocks are assumed to be rectangular and the lengths and widths of these blocks are determined in addition to their locations. Also, there is an upper and a lower bound on the aspect ratio a block may have, where the aspect ratio is the ratio of width of the block to its height. Due to its complexity, floorplanning is usually carried out in two separate steps. The first step is to determine the floorplan topology, i.e., to specify the relative positions of the blocks on a chip. The second step is then to select dimensions for each block so that the area of the enveloping rectangle is minimized while the floorplan topology determined in the first step is held fixed. The second step is usually referred to as the floorplan area minimization problem and it has been studied extensively (see, e.g. [2, 3, 4, 6, 7, 8, 9] and references therein).

Rosenberg [4] and Moh *et. al* [2] show that the floorplan area minimization problem can be formulated as a convex programming problem. Convex programming problems enjoy the property that any local solution is also global. Further, there are polynomial-time algorithms for many classes of convex programming problems.

In this paper, we show that the convex formulation of [2] can be reduced to one with the numbers of variables and constraints significantly less than those given in [2]. Specifically, we show that the number of variables can be reduced from $3n+1$ to $2n+1-p$ and the number of constraints from $5n+q$ to $4n+q-p-r$, where n denotes the number of blocks, q the number of nontrivial adjacency constraints, p the maximal number of blocks adjacent to either side of two adjacent sides of the enveloping rectangle, and r is related the slicing sub-floorplans with one-level tree structure in the floorplan. To give the exact counts of variables and constraints in our convex formulation, we develop a result related to the floorplan topology, which may be of interest in its own right. Since the complexity of solving a convex programming problem typically increases dramatically with the numbers of variables and constraints, our results thus lead to a significant reduction of computational effort in solving the floorplan area minimization problem.

The main result of the paper will be presented in Section 2, where we show how to reduce the numbers of variables and constraints from the convex formulation of [2]. We also give the result related to the floorplan topology and show how it leads to the exact counts of variables and constraints in our convex formulation. Finally, we give an example to illustrate the main results of the paper in Section 3.

2. MAIN RESULTS

Let us start with a simple floorplan area minimization problem [2]. Consider the floorplan topology in Figure 1(a). We assume that the topology has been chosen, or temporarily chosen, in the first step of the floorplan design process. For $i = 1, \cdots, 4$, Cell i shall contain one block with a given area a_i and the block also needs to satisfy the aspect ratio constraint

$$M_i w_i \geq z_i \geq m_i w_i$$

where w_i and z_i denote the width and height of the block respectively, and $M_i \geq m_i$ are positive constants. The floorplan area minimization problem is then to find the width and height of each block such that the area of the enveloping rectangle (namely, the smallest rectangle containing all the cells) is minimized. Moh *et. al* [2] show that a formulation of the above problem is

$$
(P) \quad \min\{x_2 y_3 : \quad
\begin{aligned}[t]
& w_i z_i = a_i, \\
& M_i w_i \geq z_i \geq m_i w_i, \; i = 1, \cdots, 4, \\
& x_1 \geq w_1, \\
& y_1 \geq z_1, \\
& x_2 - x_1 \geq w_2, \\
& y_2 \geq z_2, \\
& x_1 \geq w_3, \\
& y_3 - y_1 \geq z_3, \\
& x_2 - x_1 \geq w_4, \\
& y_3 - y_2 \geq z_4, \\
& y_2 - y_1 \geq 0\}
\end{aligned}
$$

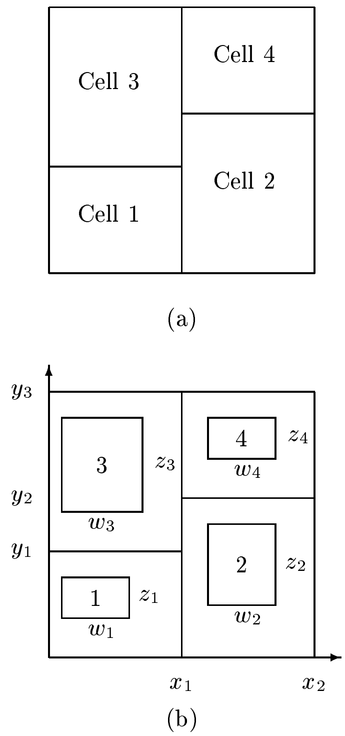


Figure 1. A simple example

where the new variables x_i's and y_i's are defined in Figure 1(b) and the minimization is over x_i's, y_i's, z_i's, and w_i's. The constraints in (P) can be categorized into four types: area constraints ($w_i z_i = a_i$), aspect ratio constraints ($M_i w_i \geq z_i \geq m_i w_i$), containment constraints ($x_1 \geq w_1, \cdots, y_3 - y_2 \geq z_4$), and adjacency constraints ($y_2 - y_1 \geq 0$). They also show that applying the change of variable $\theta \leftrightarrow \exp(\bar{\theta})$ to all variables would transform (P) into an equivalent convex optimization problem. Appendix A summarizes the key steps in showing the convexity of the transformed problem.

Now we show how to reduce the numbers of variables and constraints. To this end, we will employ the following result which gives the equivalence of four sets in a certain sense.

Proposition 1. Let $a > 0$ and $M > m > 0$ and define $\alpha = \sqrt{a/M}$ and $\beta = \sqrt{am}$. Consider the sets

$$S_1 = \{(x, y, w, z) : wz = a,\ Mw \geq z \geq mw,\ x \geq w,\ y \geq z\}$$
$$S_2 = \{(x, y, w) : wy \geq a,\ x \geq w \geq \alpha,\ y \geq \beta\}$$
$$S_3 = \{(x, y, z) : xz \geq a,\ x \geq \alpha,\ y \geq z \geq \beta\}$$
$$S_4 = \{(x, y) : xy \geq a,\ x \geq \alpha,\ y \geq \beta\}$$

Then the following statements are true.

1. $(x, y, w, z) \in S_1$ implies both $(x, y, w) \in S_2$ and $(x, y, z) \in S_3$.
2. Either $(x, y, w) \in S_2$ or $(x, y, z) \in S_3$ implies $(x, y) \in S_4$.
3. $(x, y) \in S_4$ implies $(x, y, w, z) \in S_1$ for some w and z.

Proof: It is straightforward to verify the first two statements. To show the third statement, let $w = \min\{x, \sqrt{a/m}\}$ and $z = a/w$. It is clear that $x \geq w$ and $wz = a$. Case I: $x \leq \sqrt{a/m}$. Thus $w = x$. It is easy to see that $x \geq \alpha$ is equivalent to $Mx \geq a/x$ and that $x \leq \sqrt{a/m}$ is equivalent to $a/x \geq mx$. Hence, $Mw = Mx \geq a/x = a/w = z$ and $z = a/x \geq mx = mw$. Also, $y \geq a/x = a/w = z$. Case II: $x > \sqrt{a/m}$. Thus $w = \sqrt{a/m}$ and $z = \beta$. Then $Mw \geq z = mw$ and $y \geq z$. $\square$

To simplify the notation, in the sequel we define

$$\alpha_i = \sqrt{\frac{a_i}{M_i}} \quad \text{and} \quad \beta_i = \sqrt{a_i m_i} \quad \text{for all } i$$

Consider the set of constraints in (P) involving w_1 or z_1, namely,

$$\{w_1 z_1 = a_1, M_1 w_1 \geq z_1 \geq m_1 w_1, x_1 \geq w_1, y_1 \geq z_1\} \quad (1)$$

In view of Proposition 1, it is easy to see that using the equivalence between S_1 and S_4, the set (1) can be simplified to

$$\{x_1 y_1 \geq a_1, x_1 \geq \alpha_1, y_1 \geq \beta_1\} \quad (2)$$

without affecting the optimization problem (P). Therefore, the variables w_1 and z_1 can be removed and the number of constraints can be reduced by two. Also, all the constraints in the reduced set (2) are ready to be transformed to some equivalent convex constraints using the change of variable $\theta \leftrightarrow \exp(\bar{\theta})$ (see Appendix A). Similar arguments may be applied to the set of constraints involving w_2 or z_2 and the set involving w_3 or z_3. Finally consider the set of constraints in (P) involving w_4 or z_4, namely,

$$\{w_4 z_4 = a_4, M_4 w_4 \geq z_4 \geq m_4 w_4,$$
$$x_2 - x_1 \geq w_4, y_3 - y_2 \geq z_4\} \quad (3)$$

Again, in view of Proposition 1, using the equivalence between S_1 and S_4, the set (3) can be simplified to

$$\{(x_2 - x_1)(y_3 - y_2) \geq a_4, x_2 - x_1 \geq \alpha_4, y_3 - y_2 \geq \beta_4\}$$

In this case, however, the constraint

$$(x_2 - x_1)(y_3 - y_2) \geq a_4$$

cannot be transformed into any convex constraint using the change of variable $\theta \leftrightarrow \exp(\bar{\theta})$. Instead, we use the equivalence between S_1 and S_2, and have the set (3) reduced to

$$\{w_4(y_3 - y_2) \geq a_4, x_2 - x_1 \geq w_4 \geq \alpha_4, y_3 - y_2 \geq \beta_4\} \quad (4)$$

Alternatively, we may use the equivalence between S_1 and S_3, and have the set (3) reduced to

$$\{(x_2 - x_1) z_4 \geq a_4, x_2 - x_1 \geq \alpha_4, y_3 - y_2 \geq z_4 \geq \beta_4\} \quad (5)$$

Now, all constraints in (4) or (5) are ready to be transformed to some equivalent convex constraints. Consequently, only z_4 or w_4 is removed and the number of constraints is reduced by one. What makes Cell 4 distinguish itself from other cells is that it is neither adjacent to x-axis nor adjacent to y-axis. This property limits the reduction of the number of variables or constraints to one, instead of two. Notice that at this point the choice between (4) and (5) is arbitrary. However, we will see later that since Cell 4 is sharing a pair of x-variables (i.e., x_1 and x_2) with Cell

2, using (5) can further reduce the number of constraints in the overall optimization formulation. Putting all this together, the optimization problem (P) then becomes

$$
\begin{aligned}
(P_1) \quad \min\{x_2y_3 : \quad & x_1y_1 \geq a_1, \\
& (x_2 - x_1)y_2 \geq a_2, \\
& x_1(y_3 - y_1) \geq a_3, \\
& (x_2 - x_1)z_4 \geq a_4, \\
& x_1 \geq \alpha_1, \\
& y_1 \geq \beta_1, \\
& x_2 - x_1 \geq \alpha_2, \\
& y_2 \geq \beta_2, \\
& x_1 \geq \alpha_3, \\
& y_3 - y_2 \geq \beta_3, \\
& x_2 - x_1 \geq \alpha_4, \\
& y_3 - y_1 \geq z_4 \geq \beta_4, \\
& y_2 - y_1 \geq 0\}
\end{aligned}
$$

Further simplification may be obtained when two or more cells share a pair of x-variables or y-variables. This is the case for the example in Figure 1, where Cells 1 and 3 share the pair of x-variables 0 and x_1, and Cells 2 and 4 share x_1 and x_2. As the result of this property, in (P_1), the constraints $x_1 \geq \alpha_1$ and $x_1 \geq \alpha_3$ can be combined into a single constraint $x_1 \geq \max\{\alpha_1, \alpha_3\}$. Also, the constraints $x_2 - x_1 \geq \alpha_2$ and $x_2 - x_1 \geq \alpha_2$ can be combined into $x_2 - x_1 \geq \max\{\alpha_2, \alpha_4\}$. Notice that the second reduction may not be possible if earlier we had chosen (4) in lieu of (5). Thus, the number of constraints is further reduced by two and resulting optimization problem becomes

$$
\begin{aligned}
(P_2) \quad \min\{x_2y_3 : \quad & x_1y_1 \geq a_1, \\
& (x_2 - x_1)y_2 \geq a_2, \\
& x_1(y_3 - y_1) \geq a_3, \\
& (x_2 - x_1)z_4 \geq a_4, \\
& x_1 \geq \max\{\alpha_1, \alpha_3\}, \\
& y_1 \geq \beta_1, \\
& x_2 - x_1 \geq \max\{\alpha_2, \alpha_4\}, \\
& y_2 \geq \beta_2, \\
& y_3 - y_2 \geq \beta_3, \\
& y_3 - y_1 \geq z_4 \geq \beta_4, \\
& y_2 - y_1 \geq 0\}
\end{aligned}
$$

The above procedure in further reducing the number of constraints can be easily extended to a general floorplan topology as follows: the set of constraints associated with a cell (such as (1) and (3)) sharing a pair of same x-variables (resp. y-variables) with any other cell should be simplified using the equivalence between S_1 and S_4 if this will result in a set of convex constraints; otherwise it should be simplified using the equivalence between S_1 and S_3 (resp. the equivalence between S_1 and S_2). On the other hand, if a cell shares neither a pair of x-variables nor a pair of y-variables with any other cell, and if the equivalence between S_1 and S_4 is not applicable, then one may freely use the equivalence either between S_1 and S_3 or between S_1 and S_2 for the simplification.

We are now able to count the numbers of variables and constraints for both the convex formulation of [2] and ours for a general floorplan topology. Let n denote the number of blocks (or cells), n_x and n_y the numbers of x-variables and y-variables, respectively, p the number of blocks adjacent to x-axis or y-axis, and q the nontrivial adjacency constraints. Also, let k denote the number of slicing sub-floorplans with one-level tree structure. For each of such sub-floorplans, denote by r_i the number of blocks in the sub-floorplan, $i = 1, \ldots, k$. Define

$$r = \sum_{i=1}^{k} (r_i - 1)$$

Then for the convex formulation of [2], the number of variables is

$$n_x + n_y + 2n$$

and the number of constraints is

$$5n + q$$

(i.e., n area constraints plus $2n$ aspect ratio constraints plus $2n$ containment constraints plus q adjacency constraints). For our convex formulation, the number of variables is

$$n_x + n_y + n - p$$

and the number of constraints is

$$4n + q - p - r$$

(i.e., n area constraints plus $2n$ aspect ratio constraints plus $n - p$ x-axis containment constraints plus q adjacency constraints minus r due to x-variables or y-variables sharing). It is obvious to see that to further reduce the number of variables and the number of constraints, we may rotate the floorplan to have the maximal p which leads to the minimal $n_x + n_y + n - p$ and $4n + q - p - r$.

To present the main result of the paper, it remains to show

$$n_x + n_y = n + 1$$

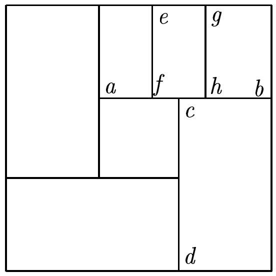
Without loss of generality, we may assume that the floorplan topology is drawn in such a way that all x-variables have distinct nominal values and all y-variables also have distinct nominal values. Notice that this assumption does not prevent any two x-variables (or y-variables) from being equal to each other at the solution.

Definition 1. Given a floorplan topology, an H-line is a line inside the enveloping rectangle with each end connected to another line perpendicular to it to form a T shape on each end. $\square$

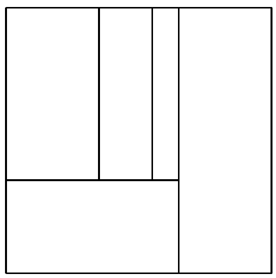
It can be easily checked that all lines inside the enveloping rectangle of a floorplan are H-lines. Given an H-line, there are two sets of lines perpendicular and connected to it; one set per side and any set may be empty. Let us denote them by T_1 and T_2, respectively. For example, in Figure 2(a), $\overline{ab}$ is an H-line with $T_1 = \{\overline{cd}\}$ and $T_2 = \{\overline{ef}, \overline{gh}\}$.

Definition 2. Given a floorplan topology G and an H-line H, the *degradation of G with respect to H*, denoted by $D(G, H)$, is the process of removing the H-line H and extending all lines in its associated sets T_1 and T_2 until touching other lines. $\square$

A degradation always leads to a new floorplan topology. For example, in Figure 2(a), performing the degradation $D(G, \overline{ab})$ leads to the new floorplan topology in Figure 2(b).



(a)



(b)

Figure 2. Floorplan G and its degradation

Notice that degradation causes both the number of blocks and the number of H-lines to decrease by one in the new floorplan topology.

Proposition 2. A floorplan topology with n blocks has $(n-1)$ H-lines.

Proof. For a floorplan topology with n blocks, we may perform degradation $(n-1)$ times with each time randomly selecting an H-line to obtain a floorplan topology with only one block. It is clear that a one-block floorplan topology has no $H-$line. It follows that the original n-block floorplan topology must have $(n-1)$ H-lines. □

Since each H-line corresponds to an x-variable or a y-variable and the enveloping rectangle needs one additional x-variable and one additional y-variable, a direct consequence of Proposition 2 is $n_x + n_y = n + 1$. We are now ready to give the exact numbers of variables and constraints in our convex formulation of the floorplan area minimization problem.

Theorem 1. The floorplan area minimization problem can be formulated as a convex programming problem with $(2n+1-p)$ variables and $(4n+q-p-r)$ constraints. □

It is clear that we have so far used only the information on the floorplan topology in reducing the optimization formulation. However, there is still room for possibly further reducing the formulation if other information about the floorplan is used. Consider the example shown in Figure 3. Following the formulation procedure previously given, we have part of the constraints look like

$$
\begin{aligned}
& y_2 \geq \beta_1 \\
& y_1 \geq z_2 \geq \beta_2 \\
& y_2 - y_1 \geq z_3 \geq \beta_3
\end{aligned}
$$

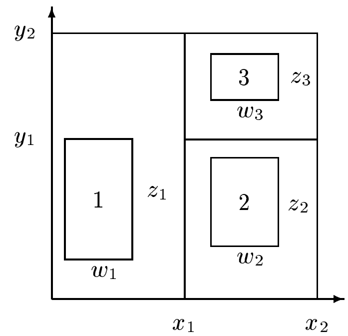


Figure 3. A further reduction example

Obviously we can see that if $\beta_2 + \beta_3 \geq \beta_1$ holds, then the constraint $y_2 \geq \beta_1$ is in fact redundant and can be removed. Such redundancy happens when a sub-floorplan with multiple-level slicing tree structure is encountered and the associated constants α's or β_i's satisfy certain inequalities (in this example, $\beta_2 + \beta_3 \geq \beta_1$). It is not difficult to see that for each such sub-floorplan, the number of possible redundant constraints is at most the number of its slicing tree structure levels minus one.

To conclude this section, we give a remark on the relation between the convex formulation in [4] and that in [2]. The formulation in [4] does not directly handle aspect ratio constraints. However, it allows constraints in the form of

$$x_i - x_j \geq \alpha_\ell \tag{6}$$

or

$$y_i - y_j \geq \beta_\ell \tag{7}$$

to be included. In view of Proposition 1, (6) and (7) together can be thought as a form of expressing aspect ratio constraints. Therefore the two formulations in [2] and [4] are in fact equivalent, although they are formed with different conceptual reasoning.

3. AN EXAMPLE

Consider the floorplan topology in Figure 3, where the number of blocks n is 23, the maximal number of blocks adjacent to either side of two adjacent sides of the enveloping rectangle p is 8, and r is 11. Thus, the corresponding floorplan area minimization problem can be formulated as a convex optimization problem with

$$2n + 1 - p = 39$$

variables and

$$4n + q - p - r = 73 + q$$

constraints. The sub-floorplan Cell *abcd* may contribute to a further reduction. Its four-level slicing tree structure may possibly result in up to 3 redundant constraints and lead to the number of constraints equal to $70 + q$. On the other hand, using the formulation in [2], the convex optimization formulation would have 70 variables and $115+q$ constraints.

APPENDIX A

Let x, y, z be variables and a a constant. Then applying the transform $\theta \leftrightarrow e^{\bar{\theta}}$ to all variables leads to the following

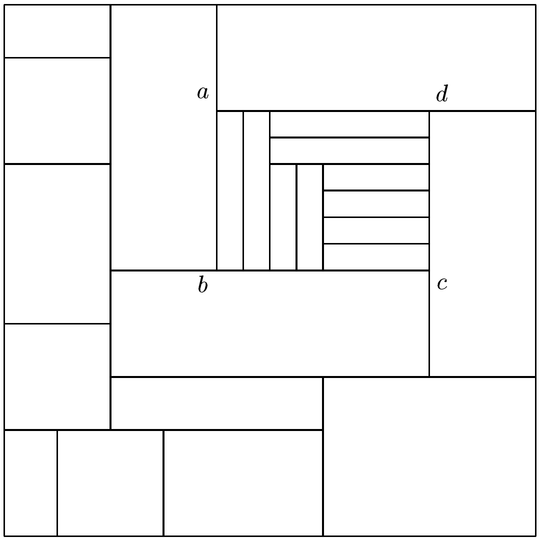


Figure 4. A general example

equivalencies

$$\begin{aligned} xy = a \quad &\Longleftrightarrow \quad \bar{x} + \bar{y} = \bar{a} \\ x \geq ay \quad &\Longleftrightarrow \quad \bar{x} \geq \bar{a} + \bar{y} \\ x - y \geq z \quad &\Longleftrightarrow \quad \log(e^{\bar{z}} + e^{\bar{y}}) - \bar{x} \leq 0 \\ (x-y)z \geq a \quad &\Longleftrightarrow \quad \log\left(e^{\bar{a}-\bar{z}} + e^{\bar{y}}\right) - \bar{x} \leq 0 \end{aligned}$$

where $\bar{a} = \log a$. Furthermore, the set of new variables that satisfy each of the inequalities on the right hand side is convex.

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