Shaping a VLSI Wire to Minimize Delay
Using Transmission Line Model

Youxin Gao and D.F. Wong
Department of Computer Sciences
University of Texas at Austin
Austin, Texas 78712

Abstract
In this paper, we consider continuous wire-sizing optimization for non-uniform wires. Our objective is to find the shape function of a wire which minimizes delay. This problem has been studied recently under the Elmore delay model. However, it is well known that Elmore delay is only a rough estimate of the actual delay and thus more accurate models should be used to determine the wire shape function. Our study uses the transmission line model which gives a very accurate estimate of the actual delay. Since previous studies under Elmore delay model suggest that exponential wire shape is effective for delay minimization, we restrict the wire shape function to be of the form \( f(x) = ae^{-bx} \). By solving the diffusion equation, we derive the transient response in the time domain as a function of \( a \) and \( b \) for both step and ramp input. The coefficients \( a \) and \( b \) are then determined so that the actual delay (50% delay) is minimized. Our algorithm is very efficient. In all the experiments we performed, the wire shape functions can be determined in less than 1 second.

1 Introduction
As the VLSI technology has been scaled down to 0.25\( \mu \)m in recent years and is expected to be scaled down to 0.065\( \mu \)m in the near future [14], the interconnect delay becomes an important factor to achieve high performance. In deep submicron design, interconnect delay is shown to be 10 to 100 times bigger than the intrinsic gate delay for a global interconnect [3], and thus dominates the circuit delay. To reduce interconnect delay, wire-sizing is found to be an effective way. One of the approaches to wire-sizing is continuous wire-sizing. Continuous wire-sizing describes the wire by a continuous shape function. Previous studies based on Elmore delay model [4, 13] have found that optimal shape function is exponential [1, 6] or near-exponential [2, 5, 7]. However, it is well known that Elmore delay model is only a rough estimate of the actual delay. First, Elmore delay is the first moment of the impulse response and it is only an upper bound of the 50% delay for step input [8]. Its accuracy depends on the circuit topology and the input slope (e.g., step input or ramp input). Second, Elmore delay can not deal with inductance. Third, Elmore delay can not provide the transient information of the response, which becomes more and more important in deep submicron design. Therefore more accurate delay models should be used in wire-sizing optimization.

In this paper, we consider continuous wire-sizing optimization for non-uniform wires. Our objective is to find the shape function of a wire which minimizes the actual delay. Our study uses the transmission line model which gives a very accurate estimate of the actual delay. Since previous studies under Elmore delay model suggest that exponential wire shape is effective for delay minimization, we restrict the wire shape function to be of the form \( f(x) = ae^{-bx} \), where \( f(x) \) is the wire width at position \( x \). By solving the diffusion equation, we derive the transient response in the time domain as a function of \( a \) and \( b \) for both step and ramp input. The coefficients \( a \) and \( b \) are then determined so that the actual delay (50% delay) is minimized. Our algorithm is very efficient. In all the experiments we performed, the wire shape functions can be determined in less than 1 second.

We would like to point out that a continuous wire shape would not be expensive to fabricate. It is not necessary to ultra-accurately reproduce the continuous shape on the silicon, because rounding the continuous shape to the nearest available litho width will give virtually no degradation in the wire delay. The staircase shape can be stamped out just as easily as any other mask shape.

The rest of the paper is organized as follows. In section 2, we give a brief introduction to the transmission line model. Then we derive the transient response in \( s \)-domain for an ideal network without driver and load. In section 3, we extend the analysis in section 2 to include driver and load. Then the transient responses in time domain for step and ramp input are derived as the sum of residues. In section 4, we show some experimental results.

---

*This work was partially supported by the Texas Advanced Research Program under Grant No. 033895288 and by a grant from the Intel Corporation.
transmission line model allows us to include it in the analysis. We expect to add inductance in our future work. The equilibrium equations of the circuit, by Kirchhoff’s voltage and current laws, are

\[ v(t, x + \Delta x) - v(t, x) \approx -r \Delta v(t, x) \]  
\[ i(t, x + \Delta x) - i(t, x) \approx -c \frac{\partial v(t, x + \Delta x)}{\partial t} \]

By taking the limit as \( \Delta x \to 0 \), we obtain the following partial differential equations.

\[ \frac{\partial v(t, x)}{\partial x} = -r(x) i(t, x) \]  
\[ \frac{\partial i(t, x)}{\partial x} = -c(x) \frac{\partial v(x, t)}{\partial t} \]

Let \( V(s, x) \) and \( I(s, x) \) be the Laplace transforms of \( v(t, x) \) and \( i(t, x) \) respectively. According to the definition of Laplace transform, we have \( V(s, x) = \int_0^\infty v(t, x)e^{-st}dt \) and \( I(s, x) = \int_0^\infty i(t, x)e^{-st}dt \). \( V(s, x) \) and \( I(s, x) \), which are sometimes called transform voltage and current, represent the voltage and current in s-domain. Taking Laplace transform on the above equations gives us

\[ \frac{\partial V(s, x)}{\partial x} = -r(x) I(s, x) \]  
\[ \frac{\partial I(s, x)}{\partial x} = -c(x) sV(s, x) \]

Since there is no derivative of \( s \) shown in the above, we can simply replace all the partial derivatives with ordinary derivatives as follows,

\[ \frac{dV(s, x)}{dx} = -r(x) I(s, x) \]  
\[ \frac{dI(s, x)}{dx} = -c(x) sV(s, x) \]

In the above equations, \( V(s, x) \) and \( I(s, x) \) are correlated. To get separate differential equations for \( V(s, x) \) and \( I(s, x) \), we differentiate (7) and substitute it into (8), then we get

\[ \frac{dV(s, x)}{dx^2} = r(x)c(x)sV(s, x) + \frac{1}{r(x)} \frac{dr(x)}{dx} \frac{dV(s, x)}{dx} \]

Similarly, differentiate (8) and substitute into (7), we get

\[ \frac{dI(s, x)}{dx^2} = r(x)c(x)sI(s, x) + \frac{1}{c(x)} \frac{dc(x)}{dx} dI(s, x) \]

For a uniform wire, \( r(x) \) and \( c(x) \) are position independent and thus are constants. However, for a non-uniform wire, \( r(x) \) and \( c(x) \) are functions of \( x \). Since previous studies based on Elmore delay model have found that exponential wire shape is effective for delay minimization, we restrict the wire shape function to be of the form \( f(x) = ae^{-bx} \). \( r(x) \) and \( c(x) \) can be calculated as:

\[ r(x) = \frac{r_0}{a} e^{-bx} \]
\[ c(x) = c_0 e^{-ax} \]

where \( r_0 \) and \( c_0 \) are unit resistance and unit area capacitance respectively. Substitute these into (9) and (10), we get

\[ V'' - bV' - r_0 c_0 sV = 0 \]
\[ I'' + bI' - r_0 c_0 sI = 0 \]
The solutions to these ordinary differential equations are straight forward. We write the solutions as
\[
V(s,x) = e^{\frac{b}{2} \xi} \left[ A_1 e^{-\Gamma x} + A_2 e^{\Gamma x} \right]
\]
(14)
\[
I(s,x) = e^{-\frac{b}{2} \xi} \left[ B_1 e^{-\Gamma x} + B_2 e^{\Gamma x} \right]
\]
(15)
where \( \Gamma = \frac{1}{2} \sqrt{B^2 + 4\eta_0 \cos \xi} \). In terms of \( \sinh(x) \) and \( \cosh(x) \), we can rewrite them as,
\[
V(s,x) = e^{\frac{b}{2} \xi} \left[ A_1 \cosh \Gamma x + A_2 \sinh \Gamma x \right]
\]
(16)
\[
I(s,x) = e^{-\frac{b}{2} \xi} \left[ B_1 \cosh \Gamma x + B_2 \sinh \Gamma x \right]
\]
(17)
where \( A_1, A_2, B_1 \) and \( B_2 \) are integration constants. They are determined by the boundary conditions. Note that those \( A, B \) constants in (16) and (17) are actually different from those in (14) and (15). To determine the constants, let \( x = 0 \), and denote the boundary conditions at input as \( V(s,0) \) and \( I(s,0) \), then we have \( A_1 = V(s,0) \) and \( B_1 = I(s,0) \). Differentiating (16) and (17) with respect to \( x \), then evaluating at \( x = 0 \), we get
\[
\frac{dV(s,x)}{dx} \bigg|_{x=0} = \frac{1}{2} b A_1 + \Gamma A_2
\]
(18)
\[
\frac{dI(s,x)}{dx} \bigg|_{x=0} = -\frac{1}{2} b B_1 + \Gamma B_2
\]
(19)
At the same time, deriving directly from (7) and (8) gives
\[
\frac{dV(s,x)}{dx} \bigg|_{x=0} = -\frac{\eta_0}{a} I(s,0)
\]
(20)
\[
\frac{dI(s,x)}{dx} \bigg|_{x=0} = -\alpha_0 s V(s,0)
\]
(21)
Therefore we can solve for \( B_2 \) and \( A_2 \) as
\[
A_2 = \Gamma \left[ -\frac{\eta_0}{a} I(s,0) - \frac{1}{2} b V(s,0) \right]
\]
(22)
\[
B_2 = \Gamma \left[ -\alpha_0 s V(s,0) + \frac{1}{2} b I(s,0) \right]
\]
(23)
Substitute these into (16) and (17), we get transform voltage and current as follows, with all integration constants substituted by boundary conditions,
\[
V(s,x) = e^{\frac{b}{2} \xi} \left[ V(s,0) \left( \cosh \Gamma x - \frac{b}{2 \Gamma} \sinh \Gamma x \right) \right.
\]
\[
- I(s,0) \frac{\eta_0}{\alpha} \sinh \Gamma x \right]
\]
(24)
\[
I(s,x) = e^{-\frac{b}{2} \xi} \left[ I(s,0) \left( \cosh \Gamma x + \frac{b}{2 \Gamma} \sinh \Gamma x \right) \right.
\]
\[
- V(s,0) \frac{\alpha_0 s}{\Gamma} \sinh \Gamma x \right]
\]
(25)
It is convenient to express the solution in terms of \( ABCD \) parameters as follows:
\[
\begin{bmatrix} V(s,x) \\ I(s,x) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(s,0) \\ I(s,0) \end{bmatrix}
\]
(26)
where
\[
A = (\cosh \Gamma x - \frac{b}{2 \Gamma} \sinh \Gamma x) e^{\frac{b}{2} \xi}
\]
(27)
\[
B = -\frac{\eta_0}{\alpha} \frac{b}{2 \Gamma} e^{\frac{b}{2} \xi} \sinh \Gamma x
\]
(28)
\[
C = -\frac{\alpha_0 s}{\Gamma} e^{\frac{b}{2} \xi} \sinh \Gamma x
\]
(29)
\[
D = (\cosh \Gamma x + \frac{b}{2 \Gamma} \sinh \Gamma x) e^{-\frac{b}{2} \xi}
\]
(30)
It is obvious that if we let \( b = 0 \) in the above, it gives us the result for the uniform wire, which has been studied previously in [9, 10, 11]. Let \( x = L \), it gives s-domain response at the load end. However, equation (26) is derived by considering an ideal network without driver and load, thus the use of (26) is limited. In the next section, we will extend (26) to the case with driver and load added into the network.

3 Transmission Line with Driver and Load

![Figure 3: Two-port network for non-uniform wire with driver impedance \( Z_D \) and load impedance \( Z_L \). If \( V_1 \) port is short circuited, \( Z_L \) becomes the load.](image)

In this section, we derive the transient response with driver and load added into the network. The overall network can be represented by three cascaded two-port networks shown in Fig.3. To simplify the expression, let \( V_1 = V(s,0) \), \( I_1 = I(s,0) \), \( V_2 = V(s,L) \) and \( I_2 = I(s,L) \). We also denote each two-port network as a pair of voltages (\( V_{in}, V_{out} \)), where \( V_{in} \) and \( V_{out} \) represent input and output voltages respectively. Therefore, in Fig.3, these three cascaded two-port network are \( V_{D1} (V_1, V_2), V_{I1} (V_1, V_2) \) and \( V_{L1} (V_2, V_L) \). We need to find the relation between transform voltage \( V_D \) at input and \( V_L \) at output. Given equation (26), the inverse relation can be derived as
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}
\]
(31)
where
\[
A' = (\cosh \Gamma L + \frac{b}{2 \Gamma} \sinh \Gamma L) e^{-\frac{b}{2} \xi}
\]
(32)
\[
B' = \frac{\eta_0}{\alpha} \frac{b}{2 \Gamma} e^{-\frac{b}{2} \xi} \sinh \Gamma L
\]
(33)
\[
C' = -\frac{\alpha_0 s}{\Gamma} e^{-\frac{b}{2} \xi} \sinh \Gamma L
\]
(34)
\[
D' = (\cosh \Gamma L - \frac{b}{2 \Gamma} \sinh \Gamma L) e^{\frac{b}{2} \xi}
\]
(35)
For two-port network \( (V_D, V_L) \), it only consists of driver impedance. The relation between \( V_D \) and \( V_L \) is thus straightforward.
\[
\begin{bmatrix} V_D \\ I_D \end{bmatrix} = \begin{bmatrix} 1 & Z_D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}
\]
(36)
Similarly, we get the expression for two-port network \((V_2, V_L)\),
\[
\begin{bmatrix}
V_2 \\
I_L
\end{bmatrix} = \begin{bmatrix}
1 & Z_L \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_L \\
I_L
\end{bmatrix}
\] (37)

By substituting (31) and (37) into (36), we get
\[
\begin{bmatrix}
V_D \\
I_D
\end{bmatrix} = \begin{bmatrix}
1 & Z_D \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{V_L}{A'} \\
\frac{Z_L}{C'}
\end{bmatrix}
\] (38)

Let \(V_L\) be short circuited, i.e., \(V_L = 0\) in the above, so that \(Z_L\) becomes the load. Then \(I_L\) is expressed as
\[
I_L = \frac{V_D}{A'Z_L + B' + Z_D\alpha'Z_L + Z_D\beta'}
\] (39)

Similarly, \(V_L = 0\) in (37) gives \(V_L = Z_LI_L\). Finally, we get
\[
V_2 = V_D \frac{Z_L}{A'Z_L + B' + Z_D\alpha'Z_L + Z_D\beta'}
\] (40)

Equation (40) shows the relation between transition voltage \(V_D\) and \(V_2\), i.e., the relation in \(s\)-domain. To get the voltage response in time domain, we need to find the inverse Laplace transform of (40).

**3.1 Transient Response for Step Input**

First, we consider the case where the driver voltage source has a step input, i.e., \(v_D(t) = U(t)\), where \(U(t)\) is step function. The Laplace transform of \(v_D(t)\) is thus \(V_D(s) = \frac{1}{s}\). In \(s\)-domain, the driver impedance and load impedance are \(Z_D = R_D\) and \(Z_L = \frac{1}{sC_L}\) respectively, where \(R_D\) is the driver resistance and \(C_L\) is the load capacitance. Equation (40) becomes,
\[
V_2 = \frac{1}{s} \frac{\Gamma e^{\frac{b}{a}t}}{M \cosh \Gamma L + N \sinh \Gamma L}
\] (41)

where
\[
M = \Gamma (R_D C_L e^{\frac{b}{a}t} + 1)
\] (42)
\[
N = \frac{b}{2} + \frac{r_0}{a} C_L e^{\frac{b}{a}t} + R_D \cos \alpha s
\] (43)

In (41), although we have \(\Gamma = \frac{1}{2} \sqrt{b'^2 + 4r_0 \cos s}\) which is square root of \(s\), the only singularities are poles. There is no branch point. The poles are at \(s = 0\) and the roots of the following transcendental equation
\[
M \cosh \Gamma L + N \sinh \Gamma L = 0
\] (44)

Since we are only considering \(RC\) circuit, all the poles are real and negative. We denote the roots of (44) as \(-s_k\), where \(s_k > 0\), \(k = 1, 2, 3, \ldots\). For (41), it can be shown that Jordan’s lemma is satisfied, so the sum of residues will give the right solution to the inverse Laplace transform [15]. The voltage response in time domain is thus
\[
v_2(t) = \lim_{s \to 0} \frac{\Gamma e^{\frac{b}{a}t}}{M \cosh \Gamma L + N \sinh \Gamma L}
\] (45)

\[
+ \sum_{k=1}^{\infty} \frac{\Gamma e^{-s_k t}}{s_k (M \cosh \Gamma L + N \sinh \Gamma L)}
\]

\[
= \left(1 - \sum_{k=1}^{\infty} \frac{\Gamma e^{-s_k t}}{s_k (M \cosh \Gamma L + N \sinh \Gamma L)}\right) + \lim_{s \to 0} \frac{\Gamma e^{\frac{b}{a}t}}{M \cosh \Gamma L + N \sinh \Gamma L}
\] (45)

where
\[
M_1 = \frac{1}{2t} r_0 \cos \left(-R_D C_L s_k e^{\frac{b}{a}t} + 1\right) + \Gamma R_D C_L e^{\frac{b}{a}t}
\]
\[
+ \frac{L}{2t} r_0 \cos \left(\frac{b}{2} + \frac{r_0}{a} C_L s_k e^{\frac{b}{a}t} - R_D \cos \alpha s\right)
\]
\[
+ \frac{R_D}{2} b C_L s_k e^{\frac{b}{a}t}
\] (46)

\[
N_1 = \frac{L}{2} r_0 \cos \left(-R_D C_L s_k e^{\frac{b}{a}t} + 1\right) + \frac{r_0}{a} C_L e^{\frac{b}{a}t}
\]
\[
+ R_D \alpha a - \frac{R_D}{2} b C_L e^{\frac{b}{a}t}
\]

In fact, when we substitute the root \(-s_k\) into (45), \(\Gamma = \frac{1}{2} \sqrt{b'^2 + 4r_0 \cos s}\) may be complex if \(s = -s_k < \frac{-b'}{2r_0}\). As a result, \(M\) and \(N\) are also complex. Although the final voltage response will not be complex, since the complex sign \(i\) will be eliminated eventually, it would be better to rewrite the formula in this case. We define a new \(\Gamma'\) so that \(\Gamma = \Gamma' i = i \frac{1}{2} \sqrt{b'^2 + 4r_0 \cos s}\). As a result, in equation (45), functions like \(\sinh \Gamma L\) and \(\cosh \Gamma L\) associated with \(\Gamma\) will be substituted by \(i \sin \Gamma' L\) and \(\cos \Gamma' L\). With the complex symbol \(i\) cancelled out from both numerator and denominator in (45), the time domain response \(v_2(t)\) is now written as
\[
v_2(t) = 1 - \sum_{k=1}^{\infty} \frac{\Gamma e^{-s_k t} e^{\frac{b}{a}t}}{s_k (M_2 \cosh \Gamma' L + N_2 \sinh \Gamma' L)}
\] (48)

where
\[
M_2 = \frac{1}{2t} r_0 \cos \left(-R_D C_L s_k e^{\frac{b}{a}t} + 1\right) + \Gamma' R_D C_L e^{\frac{b}{a}t}
\]
\[
- \frac{L}{2t} r_0 \cos \left(\frac{b}{2} + \frac{r_0}{a} C_L s_k e^{\frac{b}{a}t} - R_D \cos \alpha s\right)
\]
\[
+ \frac{R_D}{2} b C_L s_k e^{\frac{b}{a}t}
\] (49)

\[
N_2 = \frac{L}{2} r_0 \cos \left(-R_D C_L s_k e^{\frac{b}{a}t} + 1\right) + \frac{r_0}{a} C_L e^{\frac{b}{a}t}
\]
\[
+ R_D \alpha a - \frac{R_D}{2} b C_L e^{\frac{b}{a}t}
\] (50)

(45) and (48) together give the time domain response under step input, where (45) works for \(0 \leq t \leq -\frac{b'}{2r_0}\) and (48) works for \(s \leq -\frac{b'}{2r_0}\).

**3.2 Transient Response for Ramp Input**

Now we consider the case where the driver voltage has a finite ramp input shown in Fig.4. The finite ramp function can actually be decomposed into two shifted infinite ramps as \(v_D(t) = [U(t) - (t - T_R)]U(t - T_R)/T_R\) where \(U(t)\) is step function. Thus the Laplace transform of the finite ramp input is \(V_D(s) = \frac{1}{s^2 T_R} (1 - e^{-sT_R})\). The \(s\)-domain response at the load end is thus
\[
v_2(t) = \frac{1}{s^2 T_R} (1 - e^{-sT_R}) \frac{\Gamma e^{\frac{b}{a}t}}{M \cosh \Gamma L + N \sinh \Gamma L}
\] (51)

To get the time domain response of \(V_2\), we will not use residue theorem directly on (51). The reason is explained later as a remark. Alternatively, the time domain response under finite ramp input is derived from the result under
infinite ramp input. For infinite ramp input, we write the s-domain response as

\[ U_2(s) = \frac{1}{s^2TR M \cosh TL + N \sinh TL} \frac{\Gamma e^{-\frac{bL}{2}}}{s^2TR} \]  

(52)

Let the time domain response under infinite ramp be \( u(t) \), then the time domain response under finite ramp is

\[
v_2(t) = \begin{cases} 
  u(t) & \text{for } t < T_R \\
  u(t) - u(t - T_R) & \text{for } t \geq T_R 
\end{cases}
\]  

(53)

In deriving \( u(t) \), we use residue theorem. We can check that Jordan's lemma is still satisfied. The poles are at \( s = 0 \) and the roots of transcendental equation (44), \( s = -s_k, k = 1, 2, 3, \ldots \). Comparing with step input, the only difference is that \( s = 0 \) is now a pole of order two instead of a simple pole. \( u(t) \) can be derived as the sum of residues

\[
u(t) = \frac{1}{TR} \left[ \frac{\Gamma e^{-\frac{bL}{2}}}{s^2TR} \right]_{s=0} + \sum_{k=1}^{\infty} \left[ \frac{\Gamma e^{-\frac{bL}{2}}}{s^2TR} \frac{1}{s^2TR} \right]_{s=s_k} \\
= \frac{2e^{-\frac{bL}{2}}}{TRb} (M_0 \cosh \frac{bL}{2} + N_0 \sinh \frac{bL}{2} ) \\
+ \sum_{k=1}^{\infty} \frac{\Gamma e^{-\frac{bL}{2}}}{s_k^2TR} (M_1 \cosh TL + N_1 \sinh TL) \\
+ \frac{1}{TR} \left[ \frac{2\rho_0 \alpha}{b^2} + t \right] 
\]  

(54)

where

\[
M_0 = \frac{\rho_0 \alpha}{b} + \frac{b}{2} R_D C_L e^{bl} + \frac{L}{2} \rho_0 \alpha \\
N_0 = \frac{L}{2} \rho_0 \alpha + \frac{\rho_0 \alpha}{a} C_L e^{bl} + \frac{R_D \rho_0 \alpha}{2} - \frac{b}{2} R_D C_L e^{bl} 
\]  

(55)

In the case when \( \Gamma \) is complex, similar to step input, \( \Gamma \), \( M_1, N_1 \), \( \sinh \) and \( \cosh \) in the above will be replaced by \( \Gamma^* \), \( M_2, N_2 \), \( \sin \) and \( \cos \) respectively.

**Remark 1** We take a different approach to get the time response of \( v_2 \) in (51). Although there is \( s^2 \) in the denominator, \( s = 0 \) is in fact a simple pole. This can be easily shown by expanding the first two terms as follows:

\[
\frac{1 - e^{-sTR}}{s^2TR} = \frac{1}{s^2TR} \left[ (sTR)^2 \frac{1}{2} + (sTR)^3 \frac{1}{3} + \cdots \right] \\
= \frac{1}{s} \left[ 1 - \frac{1}{2} T_R s + \frac{1}{3} T_R^2 s^2 + \cdots \right] 
\]

Follow the same procedure under step input, the time domain response under ramp input can be expressed as the sum of residues. So we get, for \( s_k < \frac{\rho_0^2}{4\rho_0 \alpha} \):

\[
v_2(t) = 1 + \sum_{k=1}^{\infty} \frac{1 - e^{-sTR}}{s_k^2TR} \frac{\Gamma e^{-\frac{bL}{2}}}{s_k^2TR} + M_1 \cosh TL + N_1 \sinh TL 
\]  

(57)

and, for \( s_k \geq \frac{\rho_0^2}{4\rho_0 \alpha} \):

\[
v_2(t) = 1 + \sum_{k=1}^{\infty} \frac{1 - e^{-sTR}}{s_k^2TR} \frac{\Gamma e^{-\frac{bL}{2}}}{s_k^2TR} + M_2 \cosh TL + N_2 \sinh TL 
\]  

(58)

where \( M_1, N_1, M_2 \) and \( N_2 \) have been defined in (46), (47), (49) and (50) respectively. The problem of using (57) or (58) is that it can not give the time response prior to \( t = T_R \).

4 Experimental Results

In this section, we present some experimental results. We make use of the fact that the transient response in RC circuit is monotonic. For a step input, given time domain response shown in (45) and (48), we use binary search to find the 50% delay. To find coefficients \( a \) and \( b \) which minimize the 50% delay, we use a general nonlinear programming solver, where initial values of \( a \) and \( b \) are chosen from previous studies using Elmore delay model [6, 7]. The objective function is the 50% delay which depends on \( a \) and \( b \). Our experiments show that the 50% delay is a smooth function of \( a \) and \( b \). Therefore, the nonlinear programming solver is efficient in finding the optimal solution. For illustration, a 50% delay contour map versus \( a \) and \( b \) is shown in Fig.5, where we have normalized the delay values so that the minimum delay has value 1.0.

The circuit parameters are chosen as follows: \( r_0 = 0.03Q/\mu m, c_0 = 0.2fF/\mu m^2 \). Other parameters, such as \( R_D, C_L, \) and \( L \) are subject to change. These parameters are listed in Table 1 along with the experimental results. In Table 1, columns 5-6 list the best choice of \( a, b \) coefficients which minimize the 50% delay. Column 7 gives the calculated Elmore delay for each shape function. The last column shows the calculated 50% delay. From this table, we find that Elmore delay is off the 50% delay by about 20% on average. In the worst case example 5, Elmore delay is off by about 40%.

For ramp input, we do similar experiments but pay more attention to rising time \( t_{10} \). Elmore delay for ramp input can be approximated by \( T_D = T_{10} + \frac{1}{2} t_{10} \) [11].
Table 1: Circuit parameters and experimental results under step input. $a, b$ are coefficients of shape function $f(x) = ae^{-bx}$.

<table>
<thead>
<tr>
<th>Example</th>
<th>$R_C$ (Ω)</th>
<th>$C_L$ (μF)</th>
<th>$L$ (μm)</th>
<th>$a$ (ns)</th>
<th>$b$ (ns)</th>
<th>Elmore</th>
<th>50% delay (ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
<td>3000</td>
<td>17.08</td>
<td>$1.757 \times 10^{-3}$</td>
<td>0.1291</td>
<td>0.1071</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>100</td>
<td>3000</td>
<td>24.43</td>
<td>$1.179 \times 10^{-3}$</td>
<td>0.1867</td>
<td>0.1590</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1000</td>
<td>3000</td>
<td>24.43</td>
<td>$7.083 \times 10^{-4}$</td>
<td>0.3333</td>
<td>0.2832</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>1000</td>
<td>3000</td>
<td>24.43</td>
<td>$1.454 \times 10^{-3}$</td>
<td>0.7810</td>
<td>0.5908</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>1000</td>
<td>3000</td>
<td>101.8</td>
<td>$1.692 \times 10^{-3}$</td>
<td>0.2766</td>
<td>0.1941</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>1000</td>
<td>30000</td>
<td>101.8</td>
<td>$7.708 \times 10^{-10}$</td>
<td>3.3415</td>
<td>2.6900</td>
</tr>
</tbody>
</table>

Table 2: Circuit parameters and experimental results under ramp input. $a, b$ are coefficients of shape function $f(x) = ae^{-bx}$.

<table>
<thead>
<tr>
<th>Example</th>
<th>$R_C$ (Ω)</th>
<th>$C_L$ (μF)</th>
<th>$L$ (μm)</th>
<th>$a$ (ns)</th>
<th>$b$ (ns)</th>
<th>$I_R$ (mA)</th>
<th>Elmore</th>
<th>50% delay (ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1,000</td>
<td>3,000</td>
<td>5.0</td>
<td>$1.6923 \times 10^{-3}$</td>
<td>0.5</td>
<td>0.5215</td>
<td>0.4807</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>1,000</td>
<td>30,000</td>
<td>5.0</td>
<td>$7.3316 \times 10^{-5}$</td>
<td>0.5</td>
<td>3.951</td>
<td>2.9489</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>1,000</td>
<td>300,000</td>
<td>101.8</td>
<td>$2.949 \times 10^{-6}$</td>
<td>50</td>
<td>7.0523</td>
<td>6.4730</td>
</tr>
</tbody>
</table>

$T_{ED}$ is the Elmore delay under step input. The circuit parameters together with experimental results are listed in Table 2. In Table 2, columns 5-6 give the best choice of $a$ and $b$. Column 7 lists the rising time. The last two columns show the calculated Elmore delay and the 50% delay. Again, we find the Elmore delay is off by about 10% – 20%.

In all the experiments we performed, our algorithm is efficient, and the wire shape functions can be determined in less than 1 second.

Figure 5: The calculated delay contours versus $a$ and $b$. All delay values have been normalized by the minimum delay value.

References