A New High-Order Absolutely-Stable Explicit Numerical Integration Algorithm for the Time-Domain Simulation of Nonlinear Circuits

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Abstract
A new numerical integration method for the time domain simulation of nonlinear circuits is presented. The new method does not belong to the traditional class of linear multistep methods. Consequently it is free from Dahlquist’s barriers in terms of stability and order. The new method is shown to be both A-stable and at the same time of arbitrarily high order. In addition, the method is explicit in nature and does not require matrix inversion at each time step. Examples of linear and nonlinear circuit simulation are included.

The proposed method significantly speeds up the time domain simulation of nonlinear circuits as it combines the efficiency of an explicit method with the accuracy and large step size possible with high order.

1 Introduction
The tendency towards ever larger circuits with ever more complex devices is stretching the limits of current simulation methods. Time domain simulation of large nonlinear circuits poses a particular problem as most commonly used simulators are based on implicit linear multistep numerical integration methods which have a computational cost that is an increasingly steep nonlinear function of the circuit size. At the same time decreasing circuit feature sizes and faster signal speeds are creating devices with more complex models. It is clear that some way must be found to increase computational efficiency without sacrificing any accuracy in the analysis if current simulation practices are to continue to be followed.

Explicit numerical integration methods are intrinsically faster than commonly used implicit methods. Explicit methods directly calculate the solution at each step. Implicit integration methods solve a system of equations for each solution step often requiring iterative solution methods [1, 2] to satisfy nonlinear algebraic equations. Solution of the equations is the main computational cost in the analysis of large networks. Implicit methods are commonly used in circuit simulation because of their stability. Lack of stability causes the normally efficient explicit methods to be unsuitable for circuit simulation.

Traditional linear multistep (LMS) numerical integration methods, both explicit and implicit, are based on polynomial approximations in the time domain [1, 2, 3]. In this paper a new explicit method based on rational function approximation in the time domain is proposed. When certain conditions are placed on the coefficients of the rational function the method is stable over the entire left half plane and the problems faced by previous explicit methods are avoided. In addition it will be shown the method can be of high order without sacrificing any stability properties. This allows larger step sizes to be taken than with previously available methods.

2 Numerical Integration Using Rational Function Approximations
Dahlquist [4] has shown that

1. An explicit LMS method cannot be A-stable (Dahlquist’s barrier #1),
2. A LMS algorithm that is A-Stable cannot exceed order 2 (Dahlquist’s barrier #2),
3. The trapezoidal rule is the most accurate A-stable LMS method of order 2,
4. A-stable methods of order higher than 2 exist but not in LMS form.

The first three points are commonly known and are applied in the form of step size control for LMS methods. The last point serves as the basis for the new method presented in this section which does not fall into the category of LMS methods and thus is not constrained by Dahlquist’s barriers.
Consider the initial-value problem
\[ \dot{x} = f(x, t), \quad x(t = 0) = x_0 \] (1)
where \( f() : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N \) is a vector valued function from the \((N + 1)\) dimensional \((x, t)\) Euclidean space \(\mathbb{R}^{N+1}\) into the \(N\) dimensional \(x\) space \(\mathbb{R}^N\). The proposed method provides solution to (1) using a sequence of local approximation functions \(x_i(t), t_{i-1} \leq t < t_i, i = 1, 2, 3, \ldots\). For each component \(x^k_i\) of the vector \(x\) a rational function approximation is developed at the point \(t_{i-1}\) as
\[ x^k_i(t) = \phi_{m/n}^{(k,i)}(h) = \frac{\sum_{j=0}^{m} a_j h^j}{\sum_{j=0}^{n} b_j h^j}, \quad h = t - t_{i-1} > 0 \] (2)
where \(a_j, b_j \in \mathbb{R}\) are parameters of the approximation with \(b_0 = 1\). The method is stepped in time by evaluating \(x^k_i(t_i)\) and using this as the initial condition for the next approximation \(x_{i+1}(t)\). A new set of approximations \(\phi_{m/n}(h)\) is calculated for each interval as the method is stepped.

To test the stability properties of the method consider the test equation introduced by Dahlquist [4]
\[ \frac{dx}{dt} = qx, \quad x(t = 0) = x_0 \] (3)
and applying (2) successively for \(M\) intervals yields the solution
\[ x_M = x_0[\phi(h)]^M \] (4)
where \(\phi(t)\) is a rational function approximation to \(\exp(qt)\). For the numerical method to be stable the solution \(x_M\) should approach zero. As \(M \rightarrow \infty\) this will be the case if the approximation \(\phi\) meets the definition of A-acceptability so that \(|\phi(h)| < 1\).

Detailed study of rational approximations and maximal approximants [5, 6, 7, 8, 9, 10] has shown that

**First Ehle Conjecture [7]:**

The A-acceptable maximal approximants \(\phi_{m/n} \in \Pi_{m/n}\) to \(\exp(z)\) are exactly those approximants which \(0 \leq m \leq n \leq m + 2\).

That \(m < n\) to assure A-acceptability is reasonably obvious as the asymptotic behavior of (2) as \(z \rightarrow \infty\) is \(a_m z^m / b_n z^n\). Proof of the whole conjecture was first provided in [8].

Thus the rational function \(\phi(h)\) will be A-acceptable if it is a maximal approximant to \(\exp(z)\) and the conditions on \(m\) and \(n\) are observed. Restricting the rational function to be an A-acceptable maximal approximate, maximizes its accuracy, and results in the function being a Padé approximate by definition [10]. Other choices of form for the rational function might meet the A-acceptable criteria, but they would not be maximal approximate.

### 3 Computation of the Rational Function Approximation

Methods for the construction of the required approximant are well known [10] and are based on expanding \(x(t)\) in a Taylor series as
\[ x(t) = \sum_{j=0}^{p} c_j t^j \] (5)
where \(c_j, j = 0, 1, 2, \ldots\) are the time domain moments of \(x\). Given the Taylor series of order \(p\), a rational approximation is constructed such that
\[ \phi_{m/n}(t) = \sum_{j=0}^{p} c_j t^j + \mathcal{O}(p + 1) \] (6)
where \(\phi_{m/n}(t)\) is given by (2) and the first \(p = m+n+1\) moments are matched by the rational approximation. Calculation of the coefficients \(a_j, b_j\) of the rational function approximation in terms of the moments \(c_j\) is straightforward [10] and will not be described here.

Without loss of generality, calculation of the time-domain moments required for approximating \(x_i(t)\) will be described using \(t_{i-1} = 0, x(0) = x_0\) as the point of reference. At \(t_{i-1}\), \(x\) is replaced with its Taylor series representation in (1) giving
\[ \sum_{j=1}^{\infty} c_j j! t^{j-1} = f(t, \sum_{j=0} c_j t^j), \quad c_0 = x_0 \] (7)
\[ \sum_{j=0}^{M} d_j t^j \] (8)
Equating powers of \(t\) in (8) gives
\[ c_{j+1}(j + 1) = d_j \] (9)
as a recursive formula for calculating \(c_j\).

Expansions for common functions have been derived [11] such that an implementation for generalized equations can be done by storing equations in graph form. When the \(i\)th moment is required at the root of the graph a depth first traversal is done calculating the \(i\)th moment of each branch and applying the appropriate functional relationship at each node to calculate the required moment. The moments are calculated sequentially and the algorithm was implemented using local storage at nodes where the \(i\)th moment depends on moments \(j, j < i\) to minimize recursion.

### 4 Application to Circuit Equations

Consider a nonlinear network described by the modified nodal analysis [2] formulation
\[ C \frac{dv}{dt} + Gv + F(v) = h(t) \] (10)
where $C$, $G$ are matrices that represent the linear lumped elements in the circuit as well as additional structural entries required by the MNA formulation, $F(v)$ represents the nonlinear lumped elements and $b(t)$ represents the input waveforms.

Using (7)-(10) we get

$$Gv_0 + F_0(v_0) = b_0$$

$$Cv_j \times j + Gv_{j-1} + F_{j-1} = b_{j-1}, \ j > 0$$

where $F_j$ and $b_j$ are the time domain moments of $F(v(t))$ and $b(t)$ respectively.

Equation (12) represents an explicit recursive relationship for evaluation of the time-domain moments.

It should be noted that even when simulating nonlinear networks, no iterations or LU decompositions of the circuit matrices are required during the transient solution. In addition the method is multirate and of variable order in the sense that it is not constrained to have the same order of approximation or the same step size for all of the components of the solution. It is possible to identify slow changing signals and avoid recalculating the associated rational functions [11]. In addition error criteria can be formulated as a function of order and step size.

5 Examples

5.1 Linear System

Consider the linear system

$$\dot{x} = \begin{bmatrix} -1 & 8 \\ -1 & -10 \end{bmatrix} x, \ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(13)

with the response for $x^{(1)}$ as plotted in Figure 1. The rational function approximation $\phi_{3/4}$ for $x^{(1)}$ in the interval starting at $t = 0$ is

$$x_i^{(1)}(t) = \frac{1 + 12.986t + 9.4001t^2 - 6.9813t^3}{1 + 5.9860t + 14.998t^2 - 19.202t^3 + 11.096t^4}$$

(14)

Figure 1 shows $x^{(1)}$ as calculated using $\phi_{3/4}$ and steps of $h = 0.75$. Equation (14) was used for the interval $(0, .75)$ in Figure 1, then the method was stepped, and a new approximation was used for the next interval. It should be noted that (14) could have been used for the entire interval $(0, 3)$ or even $(0, \infty)$ as a good approximation to the exact solution. Figure 2 shows (14) plotted against the exact solution for the interval $(0, 0)$, essentially solving this entire interval in one step. The rational functions used in this method are smooth and intermediate points can be calculated by direct evaluation of the function.

Figure 1: Solution of the simple linear system of (13) using different methods with a step size of $h = 0.75$.

Figure 2: A single step solution to (13) using the function of Equation (14).
5.2 A Stiff Circuit

An RC circuit was constructed with 19 resistors and 19 capacitors and the component values chosen so that the eigenvalues of the state space formulation ranged from $-4.236$ to $-0.00617$ a ratio of almost 700. The response near the source and far end of this stiff system is shown in Figure 3 as calculated using a step size of 10 seconds. $\phi_{8/8}$ functions were used for all of the waveforms. The step size used is much larger than the $2/4.236 = 0.47$ second limit beyond which the forward Euler method is unstable.

The results are practically identical to those calculated using HSPICE which required 108 iterations to cover the time period. Including the step of the 1 second rise time of the input the new method used 11 steps to solve the same interval. The new method did not require the inversion of the circuit matrix at any time point whereas implicit methods such as those used in HSPICE require at least one LU decomposition of the circuit equations at each time point.

5.3 Transistor Circuit

Figure 4 shows the response of a simple, one transistor, common emitter, inverter circuit. An Ebers-Moll model was used for the transistor.

Eleven time steps with $\phi_{4/4}$ rational functions were used for the simulation. Again no inversions of circuit matrices were required during any of these steps. HSPICE required 302 iterations to solve the same example. Each Newton’s iteration in HSPICE involved one LU decomposition of the circuit matrix.

5.4 Diode Circuit Example

Figure 5 and Figure 6 show a nonlinear circuit and its response. The response shown was calculated using $\phi_{8/8}$ approximations. No time points other than the break points of the input pulse, shown in Figure 7 were required (shown in the figure by vertical dashed lines). Again the results agreed with HSPICE and the proposed method required no inversion of the circuit matrices during the transient solution. HSPICE required 307 iterations to solve this circuit.
6 Conclusion
A new explicit numerical integration technique for the time-domain simulation of nonlinear networks has been presented. The method was shown to be of arbitrarily high-order while preserving A-stability.

The new method can perform nonlinear time domain circuit simulation without inverting circuit matrices or doing Newton's iterations. Avoiding these traditionally CPU intensive operations gives the method speed advantages over previously used techniques. In addition the method is of high-order allowing large step sizes and can be used with multirate techniques to further increase its efficiency.

The new method is not a linear multistep method and is a departure from techniques that have become entrenched in popular simulators such as SPICE. The techniques associated with the presented method offer new opportunities in the development of circuit simulators.

References


