Efficient Methods for Simulating Highly Nonlinear Multi-Rate Circuits

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Abstract

Widely-separated time scales appear in many electronic circuits, making traditional analysis difficult or impossible if the circuits are highly nonlinear. In this paper, an analytical formulation and numerical methods are presented for treating strongly nonlinear multi-rate circuits effectively. Multivariate functions in the time domain are used to capture widely separated rates efficiently, and a special partial differential equation (the MPDE) is shown to relate the multivariate forms of a circuit's signals. Time-domain and mixed frequency-time simulation algorithms are presented for solving the MPDE. The new methods can analyze circuits that are both large and strongly nonlinear. Compared to traditional techniques, speedups of more than two orders of magnitude, as well as improved accuracy, are obtained.

1 Introduction

Consider a fast (e.g., 1Ghz) pulse train mixed with a slow (1Khz) sinusoid, or the same pulse train but with a slowly-varying duty cycle. These are instances of *multi-rate signals*, signals that contain components varying at different, widely separated, rates. Such signals arise in diverse circuits and systems, ranging from wireless/RF systems, VCOs, PLLs and FM discriminators to switched-capacitor filters, Σ-Δ modulators, switching power converters and chirp circuits. Circuits like these are typically difficult or impossible to analyze in detail; available methods are effective for either the strongly nonlinear single-rate problem or the weak to moderately nonlinear multi-rate problem, but not for the combination of strong nonlinearities and multi-rate signals.

In this paper, a new formulation and related numerical algorithms are presented for solving a wide variety of multi-rate circuits efficiently. In particular, the approach is effective for strongly nonlinear problems. The key idea is that a multi-rate waveform can be represented much more efficiently as a function of several "time" arguments (i.e., as *multivariate functions*) than as a function of a single time argument. The concept is illustrated in Section 2 with an example. Using multivariate functions changes the circuit's differential algebraic equations (DAEs) into multi-rate partial differential equations (MPDEs). By choosing the boundaries of the MPDE appropriately, different types of multi-rate solutions, including quasi-periodic, envelope-modulated, and varying-frequency (chirp, FM, VCO-type) signals, can be found. Time-domain algorithms are used to solve the MPDE system in such a way that the efficiency of the multivariate representation is exploited. Once the multivariate functions for the circuit have been computed, information at any time-scale of interest is available directly; alternatively, time-domain waveforms and frequency-domain spectra can be generated as cheap post-processing steps.

Three numerical methods, *Multivariate FDTD*[1] (MFDTD), *Hierarchical Shooting* (HS) and *Multivariate Mixed Frequency Time* (MMFT), are presented for solving the MPDE system. MFDTD and HS are purely time-domain algorithms applicable to arbitarily nonlinear circuits. They differ mainly in their computation and memory requirements. MMFT uses a combination of frequency and time domain approaches, useful for circuits like SC filters and switching mixers where some signal paths are only mildly nonlinear while others are strongly nonlinear. All three methods handle the effects of strong nonlinearities (like spikes and pulses) efficiently by sampling the multivariate functions adaptively at non-uniformly spaced points. Iterative linear algebra techniques are especially well suited to the new algorithms on account of diagonally-dominant Jacobian matrix structure. Their use results in linear computation and memory growth with circuit size.

Previous approaches for multi-rate simulation suffer from limitations when applied to strongly nonlinear circuits. Straightforward transient integration or shooting (e.g., [1, 2]) is computationally expensive, and can be inaccurate, because fast-varying signal components have to be followed long enough to obtain information about slowly-varying ones. Harmonic Balance (HB) (e.g., [3, 4]) is inappropriate because sharp features like spikes/pulses, generated by strong nonlinearities, are not represented efficiently in a Fourier basis. Also, strong nonlinearities destroy diagonal dominance in the HB Jacobian matrix, hence existing preconditioned iterative techniques [5, 4], needed for solving large circuits, can become ineffective. The methods presented in this work alleviate the efficient representation problem (by not relying on Fourier series) and the diagonal dominance problem (the dominance of the time-domain Jacobian is not affected by the degree of nonlinearity). The mixed frequency-time method of Kundert et al [6], while applicable to a class of strongly nonlinear circuits, is limited to one strongly nonlinear rate and also to quasi-periodic signals. The MMFT method of this work can solve for general multi-rate signals with several strongly nonlinear components.

[1] FDTD = Finite Difference Time Domain.

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This work establishes that the MPDE formulation (see Section 2) is natural for solving the general multi-rate problem in the time domain. The formulation itself was first used by Brachtendorf [7] in an elegant derivation of quasi-periodic harmonic balance[2]. In the present work, the MPDE is not used purely for harmonic balance; instead, new time-domain methods capable of handling strong nonlinearities effectively are developed. In addition, boundary conditions are described that enable more general multi-rate signals than simply quasi-periodic to be analyzed.

In Section 2, multivariate signals and the MPDE formulation are illustrated and useful theorems stated. Section 3 contains a description of the numerical methods. Results demonstrating significant speedups are presented in Section 4.

2 Multi-variate Signals and the MPDE

Consider a simple two-tone quasi-periodic signal given by:

$$b(t) = \sin\left(\frac{2\pi}{T_1}t\right)\sin\left(\frac{2\pi}{T_2}t\right), \quad T_1 = 1\text{ms},\ T_2 = 0.01\text{ms} \tag{1}$$

The two tones are at frequencies $f_1 = \frac{1}{T_1} = 1\text{kHz}$ and $f_2 = \frac{1}{T_2} = 100\text{kHz}$. As seen in Figure 1, there are 100 fast-varying sinusoids of period $T_2 = 0.01\text{ms}$ modulated by a slowly-varying sinusoid of period $T_1 = 1\text{ms}$.

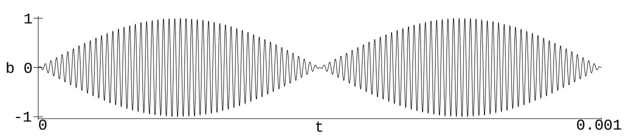

Figure 1: Example 2-tone quasi-periodic $b(t)$

In a traditional transient or shooting analysis of the circuit, the time-steps need to be spaced closely enough that each fast sinusoid in $b(t)$ is represented accurately. To generate Figure 1, 15 points were used per sinusoid, hence the total number of samples was 1500.

Now consider a *bi-variate* representation for $b(t)$ obtained as follows: for the slowly-varying parts of the expression for $b(t)$, t is replaced by t_1; for the fast-varying parts, by t_2. The resulting function of two arguments is denoted by $\hat{b}(t_1, t_2)$:

$$\hat{b}(t_1, t_2) = \sin\left(\frac{2\pi}{T_1}t_1\right)\sin\left(\frac{2\pi}{T_2}t_2\right) \tag{2}$$

Note that it is easy to recover $b(t)$ from $\hat{b}(t_1, t_2)$, simply by setting $t_1 = t_2 = t$. Note also that $\hat{b}(t_1, t_2)$ is *bi-periodic*, i.e., periodic with respect to both t_1 and t_2: $\hat{b}(t_1 + T_1, t_2 + T_2) = \hat{b}(t_1, t_2)$. The plot of $\hat{b}(t_1, t_2)$ on the rectangle $0 \le t_1 \le T_1$, $0 \le t_2 \le T_2$ is shown in Figure 2. Because $\hat{b}$ is bi-periodic, this plot repeats over the rest of the t_1-t_2 plane.

Note further that $\hat{b}(t_1, t_2)$ does not have many undulations, unlike $b(t)$ in Figure 1. Hence it can be represented by

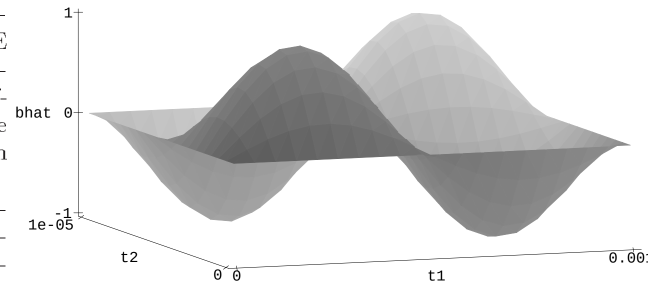

Figure 2: Corresponding 2-periodic bivariate $\hat{b}(t_1, t_2)$

relatively few points. Figure 2 was plotted with 225 samples on a uniform 15×15 grid – an order of magnitude fewer than for Figure 1.

Moreover, it is easy to generate the original quasi-periodic signal $b(t)$ from $\hat{b}(t_1, t_2)$ by using the relation $b(t) = \hat{b}(t, t)$ and the fact that $\hat{b}$ is bi-periodic: given any value of t, the arguments to $\hat{b}$ are given by $t_i = t \bmod T_i$. For example:

$$\begin{aligned} b(1.952\text{ms}) &= \hat{b}(1.952\text{ms}, 1.952\text{ms}) \\ &= \hat{b}(T_1 + 0.952\text{ms}, 195T_2 + 0.002\text{ms}) \\ &= \hat{b}(0.952\text{ms}, 0.002\text{ms}) \end{aligned}$$

The above illustrates two important features: 1. *the bi-variate form requires far fewer points to represent numerically than the original quasi-periodic signal*, yet 2. *it contains all the information needed to recover the original signal completely.* This is true not only for signals with a compact frequency-domain representation (such as $b(t)$ in Equation 1, with only two frequency components, $f_2 \pm f_1$), but also for those that cannot be represented efficiently in the frequency domain (see Figure 4 for an example).

The key to the techniques in this paper is to solve for the multivariate forms of all the node voltages and branch currents of a circuit directly. The basic notion is to re-write the circuit's differential equations in terms of the multivariate forms of the circuit's variables and to solve these equations efficiently. The multivariate forms satisfy a partial differential equation, the Multirate Partial Differential Equation (MPDE), closely related to the circuit's differential equations. By applying boundary conditions to the MPDE and solving it numerically with time-domain or mixed frequency-time methods, the desired multivariate solutions are obtained. Different types of multirate signals can be handled by choosing the boundaries appropriately.

The MPDE is based on the circuit's equations, which can always be expressed in DAE form:

$$\dot{q}(x) = f(x) + b(t) \tag{3}$$

All variables (except the time t) are vector-valued. $x(t)$ are the unknowns in the circuit (node voltages and branch currents); q the charge terms and f the resistive terms; $b(t)$ is

[2] This work came to the author's attention after he discovered the MPDE formulation independently.

the vector of excitations to the circuit (typically from independent voltage/current sources).

If there are m separate rates of change in the circuit's signals, the multivariate forms of $x(t)$ and $b(t)$ can be denoted by $\hat{x}(t_1, \cdots, t_m)$ and $\hat{b}(t_1, \cdots, t_m)$. The MPDE corresponding to Equation 3 is defined to be:

$$\frac{\partial q(\hat{x})}{\partial t_1} + \cdots + \frac{\partial q(\hat{x})}{\partial t_m} = f(\hat{x}) + \hat{b}(t_1, \cdots, t_m) \tag{4}$$

The MPDE has a number of properties that make it useful for analyzing multi-rate problems. The following theorem states that solutions to the circuit's DAEs are available on "diagonal" lines along the MPDE's multivariate solutions[3].

Theorem 1 (MPDE-DAE Relation) *If* $\hat{x}(t_1, \cdots, t_m)$ *and* $\hat{b}(t_1, \cdots, t_m)$ *satisfy the MPDE in Equation 4, then* $x(t) = \hat{x}(t + c_1, \cdots, t + c_m)$ *and* $b(t) = \hat{b}(t + c_1, \cdots, t + c_m)$ *satisfy the circuit's DAE in Equation 3, for any fixed* $c_1, \cdots, c_m$.

To solve the MPDE, it is first necessary to specify boundary conditions (BCs). Different boundary shapes and BCs lead to different types of multi-rate behaviour. Theorems 2 and 3 deal with the quasi-periodic case, which corresponds to periodic BCs on rectangular boundaries.

Theorem 2 (MPDE Sufficiency Condition) *Given an m-periodic* $\hat{b}$ *(i.e.,* $\hat{b}(t_1 + T_1, \cdots, t_m + T_m) = \hat{b}(t_1, \cdots, t_m)$*) such that* $b(t) = \hat{b}(t + c_1, \cdots, t + c_m)$*, any solution* $\hat{x}$ *of the MPDE with m-periodic boundary conditions generates an m-tone quasi-periodic solution* $x(t)$ *of the circuit DAE through* $x(t) = \hat{x}(t + c_1, \cdots, t + c_m)$*, for arbitrary* $c_1, \cdots, c_m$.

Theorem 3 (MPDE Necessity Condition) *If a quasi-periodic solution* $x(t)$ *of the circuit DAE exists for a quasi-periodic excitation* $b(t)$*, then for any* $c_1, \cdots, c_m$*, there exist m-periodic functions* $\hat{b}(t_1, \cdots, t_m)$ *and* $\hat{x}(t_1, \cdots, t_m)$ *satisfying the MPDE, and also satisfying* $x(t) = \hat{x}(t+c_1, \cdots, t+c_m)$*,* $b(t) = \hat{b}(t + c_1, \cdots, t + c_m)$.

Envelope-modulated solutions of the DAE can be obtained from the MPDE by using a mixture of initial and periodic boundary conditions:

Theorem 4 (Uniqueness of Envelope) *Assume that the circuit DAE has a unique solution given an initial condition. Then the solution* $\hat{x}(t_1, \cdots, t_m)$ *of the MPDE is unique, given the following mixed initial and periodic boundary conditions:*

$$\hat{x}(t_1, t_2 + T_2, \cdots, t_m + T_m) = \hat{x}(t_1, \cdots, t_m)$$
$$\hat{x}(0, t_2, \cdots, t_m) = h(t_2, \cdots, t_m)$$

where h *is any given initial-condition function, defined on* $\prod_{i=2}^{m}[0, T_i]$.

Varying-rate solutions (i.e., chirp/FM/VCO-type) can also be found by selecting boundaries appropriately. If the variation of rate is known a-priori, as in chirp-type signals, using periodic or initial/periodic boundary conditions on *non-rectangular boundaries* of known shape generates solutions of the desired type. For example, the boundaries leading to a linear chirp between two frequencies is illustrated for the 2-rate case in Figure 2. The frequency of the signal changes linearly from a high value to a lower one during the sloping segment of the upper boundary. VCO/FM-type signals, where the rate variation is not known before solution of the problem, can be captured by *moving or free boundaries* [9, 10] for the MPDE.

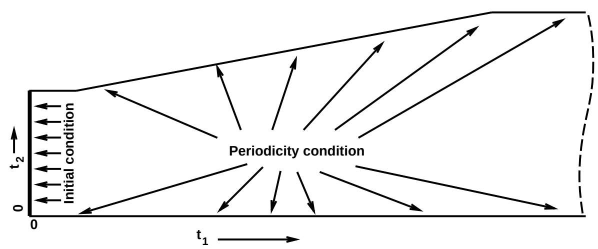

Figure 3: Non-rectangular boundaries for a chirp signal

3 Numerical Methods for the MPDE

In this section, three numerical methods are presented for solving the MPDE. Two of the methods (MFDTD and HS) solve the MPDE purely in the time domain. The MMFT method solves for some of the dimensions of the MPDE in the frequency domain and others in the time domain.

3.1 The Multivariate Finite Difference Time Domain (MFDTD) method

In this method, the MPDE of Equation 4 is solved on a grid in the $t_1, \cdots, t_m$ space. Let the grid be the set of points $\{\bar{t}_1, \cdots, \bar{t}_n\}$, where each $\bar{t}_i = (t_{1_i}, \cdots, t_{m_i})$. The partial differentiation operators of the MPDE are discretized and the MPDE collocated on this grid. This leads to a set of nonlinear algebraic equations in the unknowns $\{\hat{x}(\bar{t}_1), \cdot, \hat{x}(\bar{t}_n)\}$. The nonlinear equations are solved by a Newton-Raphson method.

For concreteness, consider the quasi-periodic 2-rate case (the envelope and varying-rate cases differ only in some boundary equations). The MPDE simplifies to:

$$\frac{\partial q(\hat{x})}{\partial t_1} + \frac{\partial q(\hat{x})}{\partial t_2} = f(\hat{x}) + \hat{b}(t_1, t_2) \tag{5}$$

with boundary conditions $\hat{x}(t_1 + T_1, t_2 + T_2) = \hat{x}(t_1, t_2)$. Consider a uniform grid $\{\bar{t}_{i,j}\}$ of size $n_1 \times n_2$ on the rectangle $[0, T_1] \times [0, T_2]$.

The equations to be solved by Newton-Raphson are obtained by first discretizing the partial differentiation operators using (for example) the Backward Euler rule, and then collocating Equation 5 on the grid. The collocation leads to a system of $n = n_1 \times n_2$ equations. The n equations are, however, in a greater number of unknowns; $n_1 + n_2$ extra unknowns $\{\hat{x}(\bar{t}_{-1,j})\}$ and $\{\hat{x}(\bar{t}_{i,-1})\}$ result from discretizing the differentiation operators on the $t_1 = 0$ and $t_2 = 0$ lines respectively. These unknowns are eliminated using the

[3]Proofs are omitted here for brevity and may be found in [8].

bi-periodic boundary conditions of the MPDE, which provide $n_1 + n_2$ equality relations between the unknowns at the boundaries, hence eliminate that many unknowns, resulting in a system of n equations in n unknowns. Denote this system by:

$$F(X) = 0, \quad \text{where} \tag{6}$$
$$F = [F_{1,1}, \cdots, F_{1,n_2}, F_{2,1}, \cdots, \cdots, F_{n_1,1}, \cdots, F_{n_1,n_2}]^T, \text{ and}$$
$$X = [\hat{x}(\bar{t}_{1,1}), \cdots, \hat{x}(\bar{t}_{1,n_2}), \hat{x}(\bar{t}_{2,1}), \cdots, \cdots, \hat{x}(\bar{t}_{n_1,n_2})]^T$$

$F(X) = 0$ is solved numerically by the Newton-Raphson method. For this, the Jacobian matrix of $F(\cdot)$ is required. If a one-sided differentiation rule like Backward Euler or Trapezoidal is used, the Jacobian has the block p-cyclic structure:

$$\frac{\partial F}{\partial X} = \begin{bmatrix} D_1 & & & & -L_{n_1} \\ -L_1 & D_2 & & & \\ & -L_2 & D_3 & & \\ & & \ddots & \ddots & \\ & & & -L_{n_1-1} & D_{n_1} \end{bmatrix} \tag{7}$$

Each block is itself a $n_2 \times n_2$ block-matrix with p-cyclic or diagonal structure. Each entry of these blocks is, in turn, a sparse matrix, consisting of circuit conductance and capacitance matrices. The overall Jacobian is therefore a very sparse, block-structured matrix.

The sparsity of $\frac{\partial F}{\partial X}$ makes iterative linear techniques (e.g., [11, 5, 4, 12]) attractive for solving the linear equations that arise at each Newton-Raphson step. An important property of the matrix, diagonal dominance, is a result of its block structure. This can be seen easily for the case where f and q are scalar: the matrix is diagonally dominant if 1. $q(\cdot)$ is linear (this can always be arranged in the circuit formulation) and 2. f' has the same sign as q', as is the case for stable circuits. Diagonal dominance is a desirable property because it ensures fast convergence for iterative linear solvers. For the case where f and q are vector functions, the matrix can be shown to be block-diagonally dominant under similar conditions.

In the above, a uniform grid was assumed for simplicity. In practice, the grid is non-uniform, built up adaptively by starting from a coarse grid and refining until an error criterion is met. The key properties of sparsity and diagonal dominance are maintained.

3.2 Hierarchical Shooting (HS)

A hierarchical extension of the classical shooting algorithm (e.g., [2, 13]) can also be used for solving the MPDE. The key to this method is to view the MPDE as an ordinary differential equation in *function space* variables. For concreteness, consider again the 2-rate MPDE of Equation 5. The variables $\hat{x}$, q, f and $\hat{b}$ are functions of two arguments t_1 and t_2, i.e., they are maps from $\mathcal{R}^2 \mapsto \mathcal{R}^k$ (k = circuit size). If one argument, say t_1, is fixed, then these are functions of the remaining argument t_2. Hence they can also be viewed as functions of a *single* argument t_1, that return values that are vector-valued *functions*; i.e., the multivariate functions are maps from $\mathcal{R} \mapsto \{h(\cdot) : \mathcal{R} \mapsto \mathcal{R}^k\}$. Let these maps be $Q(t_1)$, $X(t_1)$, $F(t_1)$ and $B(t_1)$[4]; in other words, $Q(t_1)$ equals the entire function $q(t_1, \cdot)$ for fixed t_1; similarly for the other variables.

[4] The function-valued variables F and X should not be confused with the vector-valued ones of the previous section.

The MPDE can then be written formally as a DAE in the function-valued variables:

$$\frac{dQ(X)}{dt_1} = F(X) + B(t_1) - \mathcal{D}_{t_2}[Q(X)] \tag{8}$$

$\mathcal{D}_{t_2}$ is an operator that differentiates the function (of t_2) that it operates on. Equation 8 can be treated as a normal DAE and discretized using (say) Backward Euler:

$$\frac{Q(X(t_{1_i})) - Q(X(t_{1_{i-1}}))}{t_{1_i} - t_{1_{i-1}}} = F(X(t_{1_i})) + B(t_{1_i}) - \mathcal{D}_{t_2}[Q(X(t_{1_i}))] \tag{9}$$

Equation 9 can be solved for $X(t_{1_i})$, which represents the function $\hat{x}(t_{1_i}, \cdot)$ at the fixed t_1-value t_{1_i}. Note that Equation 9 is itself a differential equation, owing to the differentiation (in t_2) term $\mathcal{D}_{t_2}[Q(X(t_{1_i}))]$. This DAE can be solved using shooting (or another method, e.g, uni-variate FDTD or harmonic balance) in t_2. This "inner loop" DAE of Equation 9 is solved for each time-step of the "outer loop" of Equation 8.

The size of the linear system from the above scheme is smaller, by the number of points in the t_1 dimension, than that for MFDTD. This leads to a large saving in memory. Also, the grid in hierarchical shooting is induced naturally by the time-step control of the transient analysis algorithm – special grid refinement algorithms are not needed. However, MFDTD can be more efficient than hierarchical shooting for circuits with slowly dying oscillations (e.g., high-Q circuits).

3.3 The Multivariate Mixed Frequency-Time (MMFT) method

$\hat{x}(t_1, \cdots, t_m)$ and $\hat{b}(t_1, \cdots, t_m)$ can be expanded in Fourier series in the arguments for which they are periodic. Some circuits are designed such that few Fourier components are significant if the expansion is performed about some, but not all, the periodic arguments – the "mildly nonlinear" rates. In such cases, the mild nonlinearity of these rates can be exploited using harmonic balance, while time-domain methods can be used for the others. The 2-rate case will again be used for the purpose of exposition. Equation 5 is rewritten as a Fourier series in t_1:

$$\sum_{i=-M}^{M} ij\omega_1 Q_i(t_2) e^{ij\omega_1 t_1} + \sum_{i=-M}^{M} \frac{\partial Q_i(t_2)}{\partial t_2} e^{ij\omega_1 t_1} = \sum_{i=-M}^{M} F_i(t_2) e^{ij\omega_1 t_1} + \sum_{i=-M}^{M} B_i(t_2) e^{ij\omega_1 t_1} \tag{10}$$

where $j = \sqrt{-1}$, $\omega_1 = \frac{2\pi}{T_1}$ and Q_i, F_i, B_i are the Fourier components in t_1 of $q(\hat{x}(t_1, t_2))$, $f(\hat{x}(t_1, t_2))$ and $\hat{b}(t_1, t_2)$ respectively. M is the (small) number of significant harmonics in t_1. Since the functions $e^{ij\omega_1 t_1}$ are linearly independent, the Fourier components in Equation 10 can be separated, leading to the following equation in vector form:

$$\frac{d\bar{Q}(t_2)}{dt_2} = -j\Omega_1 \bar{Q}(t_2) + \bar{F}(t_2) + \bar{B}(t_2) \tag{11}$$

where $\Omega_1 = \omega_1 \mathrm{diag}(M, \cdots, -M)$, $\bar{Q} = [Q_M, \cdots, Q_{-M}]^T$, $\bar{F} = [F_M, \cdots, F_{-M}]^T$ and $\bar{B} = [B_M, \cdots, B_{-M}]^T$.

Equation 11, being a vector DAE, is solved for a periodic solution by existing time-domain methods such as univariate shooting [2, 13] or univariate FDTD. The differentiation operator is discretized using a numerical integration scheme. This results in an inner system of frequency-domain equations that are solved by harmonic balance. An alternative method using shooting/FDTD in the inner loop and harmonic balance in the outer can also be derived, using the function-space variable concept of Section 3.2.

4 Experimental Results

In this section, results from applying the new methods are presented. Speedups of two orders of magnitude are obtained, as seen from the summary in Table 1.

Circuit	MPDE-based	1-variate shooting / transient
rectifier, MFDTD	40s	1h 21m
rectifier, HS	1m 09s	1h 36m
mixer, MMFT	25s	1h 52m

Table 1: CPU times (SPARC 20, 96MB, SunOS4.1.3)

A (strongly nonlinear) diode rectifier circuit, powered by a two-tone quasi-periodic power source $b(t)$, was simulated for a quasi-periodic solution using MFDTD. The circuit consisted of a diode followed by a parallel R-C filter combination. The power source was a train of fast pulses whose duty cycle was modulated at a much slower rate. Using $\mathsf{pulse}(\frac{t}{0.1\mu}, duty)$ to describe each pulse, the excitation was $b(t) = \hat{b}(t_1, t_2)$, where $\hat{b}$ was:

$$\hat{b}(t_1, t_2) = \mathsf{pulse}\left(\frac{t_2}{T_2}, 0.5 + 0.3\sin(\frac{2\pi t_1}{T_1})\right)$$

Here $T_1 = 1$ms, $T_2 = 1\mu$s. The bi-variate form $\hat{b}(t_1, t_2)$

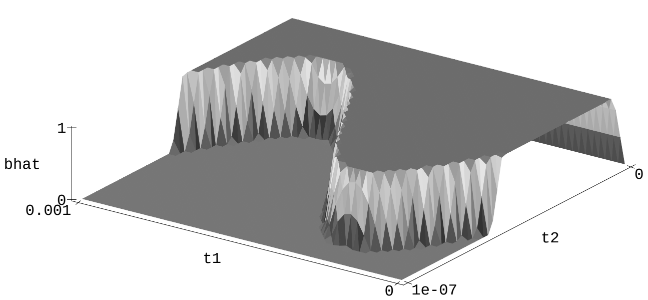

Figure 4: The bi-variate excitation $\hat{b}(t_1, t_2)$

is plotted in Figure 4. The sinusoidally-varying duty cycle is the extent of the high region while moving along the t_2 direction. The bi-variate solution $\hat{x}$ is shown in Figure 5. The low-pass filter has smoothed out the fast variations in the t_2 direction. Since the rectified output depends on the duty cycle on the input, a slow-scale sinusoidal variation is observed as a function of t_1.

The circuit was also simulated by univariate shooting for comparison. As shown in Table 1, the MFDTD method was faster by over two orders of magnitude. Plots of the univariate solution $x(t)$ are shown in Figure 6. The waveform obtained from the bi-variate solution is denoted by the legend "new", and those from univariate shooting using 20 and 50 time-steps per fast pulse by "h=T2/20" and "h=T2/50" respectively. Univariate shooting using 20 time-steps per pulse accumulated errors that grew to 15% near $t = 0.8$ms, despite tight local error control. Increasing the number of time-steps to 50 per pulse reduced the error, but it remained significant at about 3% (the CPU time in Table 1 is for this simulation).

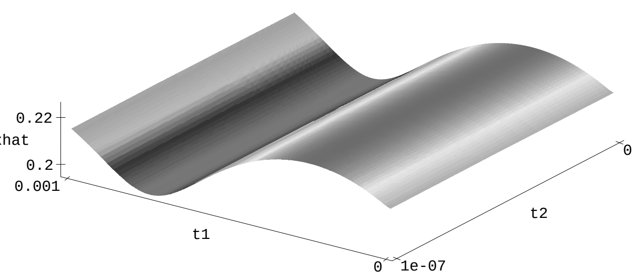

Figure 5: The bi-variate solution $\hat{x}(t_1, t_2)$

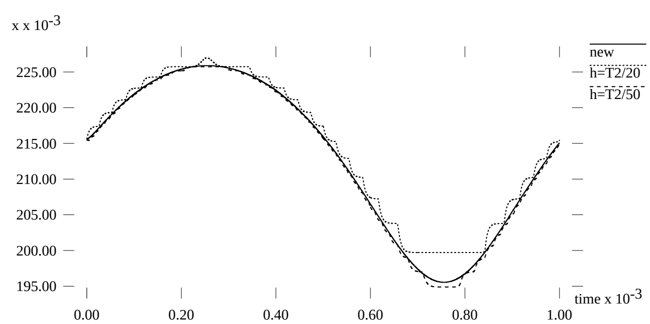

Figure 6: Rectifier output $x(t)$

The same circuit was simulated for an envelope waveform using HS. Detailed results may be found in [8]; the speed advantage of HS over univariate transient was again over two orders of magnitude.

Another circuit, a switching mixer and filter, was analyzed for intermodulation distortion using the MMFT method. The RF input to the mixer was a 100kHz sinusoid with amplitude 100mV; this sent it into a mildly nonlinear regime. The LO input was a square wave of large amplitude (1V), which switched the mixer on and off at a fast rate (900Mhz). The circuit was also simulated by univariate shooting for comparison.

Three harmonics of the RF tone $f_1 = 100\text{kHz}$ (corresponding to the t_1 variable of Section 3.3) were considered. The LO tone at $f_2 = 900\text{MHz}$ was handled by shooting in the t_2 variable. The output of the MMFT algorithm was a set of time-varying harmonics that repeated with period $T_2 = \frac{1}{f_2}$. The first and third harmonics are shown in Figure 7. The first harmonic contains information about all mix components of the form $f_1 + i f_2$, i.e., the frequencies 900.1 Mhz, 1800.1 Mhz, etc.. The main mix component of interest, 900.1 Mhz, is found by taking the first Fourier component of the waveform in Figure 7. This has an amplitude of 60mV. The third harmonic contains information about the mixes $3f_1 + i f_2$, i.e., the frequencies 900.3 Mhz, 1800.3 Mhz, etc.. The amplitude of the 900.3 Mhz component can be seen to be about 1.1mV; hence the distortion introduced by the mixer is 35dB below the desired signal.

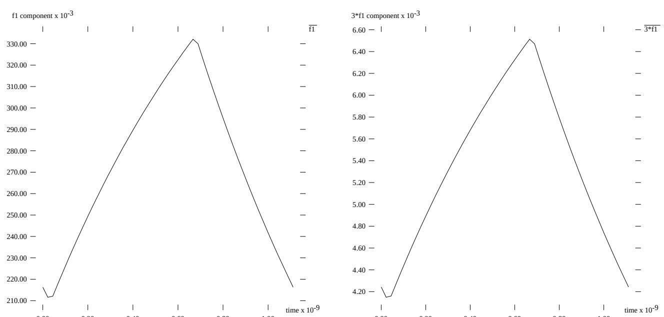

Figure 7: Mixed frequency-time output: first (left) and third (right) harmonic components

The output produced by univariate shooting is shown in Figure 8. This run, using 50 steps per fast period, took almost 300 times as long as the new algorithm (see Table 1).

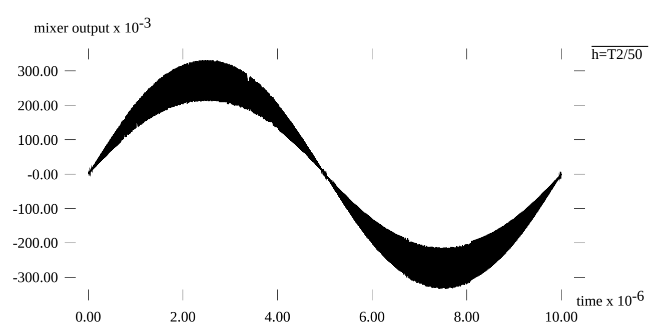

Figure 8: Mixer output from univariate shooting

5 Conclusion

A new approach for analyzing strongly nonlinear multi-rate circuits has been presented. The approach uses a MPDE formulation, with appropriate boundary conditions, to treat quasi-periodic, envelope-modulated, and changing-frequency signals. Efficiency is achieved without compromising accuracy by using adaptive time-domain numerical techniques to solve the MPDE. Two purely time-domain numerical methods are useful for strongly nonlinear circuits in general, while a third mixed frequency-time method is more suitable for circuits with some weakly nonlinear paths. The new techniques are applicable to a wide variety of circuits that are difficult or impossible to simulate with previous techniques. The methods also scale linearly with circuit size. Results demonstrating significant advantages in speed and accuracy over previous methods have been presented.

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