Least Upper Bounds on the Sizes of Symmetric Variable Order based OBDDs*

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Abstract

This paper investigates the sizes of symmetric variable order based reduced binary decision diagrams for partially symmetric Boolean functions. It gives exact bounds for the maximum number of nonterminal vertices for the cases that the set of symmetric variables is treated as block which is located either at the front or at the back of the variable order.

1. Introduction

Binary Decision Diagrams (BDDs) as a data structure for representation of Boolean functions were first introduced by Lee [5] and further popularized by Akers [1] and Moret [8]. In the restricted form of OBDDs they gained widespread application, because OBDDs are a canonical representation and allow efficient manipulations [2].

In this paper we concentrate on Boolean functions in n variables which are partially symmetric in k variables. We derive exact equations and give upper bounds for the maximum number of nonterminal vertices for the case that the block of symmetric variables are located at the top or at the bottom of the OBDD. The results generalize the least upper bounds on the size of OBDDs proven by Heap [3, 4] and Wegener [10].

The upper bounds will give the possibility to compute whether a partially symmetric Boolean function f in n variables can be represented by a fore or back symmetry ordered OBDD for given size of main storage.

2. Preliminaries

We provide a short introduction to basic notions which are important for the understanding of this paper. For the definition of reduced ordered BDDs (OBDDs) we refer to [2] and [6].

2.1. Symmetric functions

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a completely specified Boolean function and $\mathcal{V}_n = \{x_1, \ldots, x_n\}$ be the corresponding set of variables. The function f is said to be *symmetric* with respect to a set $\lambda \subseteq \mathcal{V}_n$ if f remains invariant under all permutations of the variables in λ. For completely specified functions symmetry is an equivalence relation which partitions the set $\mathcal{V}_n$ into disjoint classes $\lambda_1, \ldots, \lambda_k$ that will be named the *symmetry sets.* A function f is called *partially symmetric* if it has at least one symmetry set λ_i with size $|\lambda_i| > 1$.

2.2. Symmetry Variable Orders

In this section, we introduce the new class of symmetry variable orders introduced by Möller [7] and Panda [9]. OBBDs can be understood as partitioned into n levels labelled from the root to the leaves by 1 to n. We associate with each OBDD an array π such that $\pi[i]$ denotes the variable that corresponds to label i. The array π is called *variable order* of the OBDD.

Definition 1 Let f be a partially symmetric function with the set of symmetry sets $S = \{\lambda_1, \ldots, \lambda_k\}$. A variable order π is called a *symmetry variable order* if for each symmetry set $\lambda_i \in S$ there exists j so that $\{\pi[j], \pi[j+1], \ldots, \pi[j + |\lambda_i| - 1]\} = \lambda_i$.

By this definition, the class of symmetry variable orders consists of all variable orders where the variables of each symmetry set are located side by side. The OBDDs that correspond to symmetry orders are called *symmetry ordered* OBDDs.

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3. Bounds on the size

In the following we concentrate on Boolean functions in n variables which are partially symmetric in k variables, i.e., there are one symmetry set of size k and $n-k$ symmetry sets of size 1. Let λ denote the symmetry set of size k. We prove upper bounds for symmetry ordered OBBDs where λ is located at the front positions and for symmetry ordered OBDDs where λ is located at the back positions. We call these orders *fore symmetry order* and *back symmetry order*, respectively.

In the following we denote by W_j the maximum number of nonterminal vertices at the x_j-level of the OBDD considered of any Boolean function f with the above property.

3.1. Fore symmetry ordered OBDDs

Assume w.l.o.g. that the variable order is fixed to $x_1, x_2, \ldots, x_n$ and $\lambda = \{x_1, x_2, ..., x_k\}$ holds.

Lemma 1 The maximum number $R_{fore}(n, k)$ of nonterminal vertices of an OBDD considered is given by $\sum_{l=1}^{n} W_l$ with

- $W_l \leq min\{l, 2^{(k-l+2)2^{n-k}} - 2^{2^{n-k}}\}$ for $1 \leq l \leq k$.
- $W_l \leq min\{l, 2^{2^{n-l+1}} - 2^{2^{n-l}}\}$ for $l = k+1$
- $W_l \leq min\{2^{l-k-1} \cdot W_{k+1}, 2^{2^{n-l+1}} - 2^{2^{n-l}}\}$ for $k+2 \leq l \leq n$.

Proof: We start with $1 \leq l \leq k$. As f is partially symmetric in $x_1, \ldots, x_{l-1}$, there are $l-1+1$ cofactors with respect to $x_1, \ldots, x_{l-1}$. Thus there are at most l x_l-vertices and $W_l \leq l$ holds.

In order to prove $W_l \leq 2^{(k-l+2)2^{n-k}} - 2^{2^{n-k}}$, we compute the number of Boolean functions which are defined on $\{x_l, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ depending on variable x_l which is obviously an upper bound on W_l. The number of different assignments of $\{x_{k+1}, \ldots, x_n\}$ is 2^{n-k}. As $\{x_l, \ldots, x_k\}$ are contained in the same symmetry set, there are $k-l+2$ 'different' assignments of the variables $\{x_l, \ldots, x_k\}$. Thus, $2^{(k-l+2)2^{n-k}}$ Boolean functions are defined on $\{x_l, \ldots, x_n\}$. $2^{2^{n-k}}$ of them do not depend on $x_l, \ldots, x_k$ and are not represented by x_l-vertices. This completes the proof of the first inequation of the lemma.

Now, consider $k+2 \leq l \leq n$. The inequation $W_l \leq 2^{l-k-1} \cdot W_{k+1}$ follows by the fact that each nonterminal vertex has at most two children, so that $W_{i+1} \leq 2W_i$ holds for any i. The inequation $W_l \leq 2^{2^{n-l+1}} - 2^{2^{n-l}}$ has been proven in [4], Lemma 2, and gives the number of Boolean functions which are defined on $x_l, \ldots, x_n$ which do depend on x_l.

The upper bound on W_{k+1} follows from the proofs of the cases just discussed. q.e.d.

In order to prove good upper bounds on $R_{fore}(n, k)$ we have to compute the term which determines W_l for any l. The different cases we have to consider are given by whether the turnpoint with respect to the breadth of the OBDD is located ahead of position k (case 4), at position k (case 1), at position $k+1$ (case 2) or behind position $k+1$ (case 3).

Theorem 1 The following statements hold for n large enough ($n \geq 16$).

1. If $2^{2^{n-k}} - 2^{2^{n-k-1}} \leq k \leq 2^{2^{n-k+1}} - 2^{2^{n-k}}$, then
$$W_l \leq \begin{cases} l & \text{if } l = 1, \ldots, k \\ 2^{2^{n-l+1}} - 2^{2^{n-l}} & \text{if } l = k+1, \ldots, n \end{cases}$$
2. If $2^{2^{n-k-1}-1} - 2^{2^{n-k-2}-1} \leq k < 2^{2^{n-k}} - 2^{2^{n-k-1}}$, then
$$W_l \leq \begin{cases} l & \text{if } l = 1, \ldots, k+1 \\ 2^{2^{n-l+1}} - 2^{2^{n-l}} & \text{if } l = k+2, \ldots, n \end{cases}$$
3. If $k < 2^{2^{n-k-1}-1} - 2^{2^{n-k-2}-1}$, then
$$W_l \leq \begin{cases} l & \text{if } l = 1, \ldots, k+1 \\ (k+1)2^{l-k-1} & \text{if } l = k+2, \ldots, t \\ 2^{2^{n-l+1}} - 2^{2^{n-l}} & \text{if } l = t+1, \ldots, n \end{cases}$$
with $t = n - \lfloor \log_2(n - k + \log_2(k+1)) \rfloor + \epsilon$ for some $\epsilon \in \{0, 1\}$.
4. If $k > 2^{2^{n-k+1}} - 2^{2^{n-k}}$, then
$$W_l \leq \begin{cases} l & \text{if } l = 1, \ldots, t \\ 2^{(k-l+2)2^{n-k}} - 2^{2^{n-k}} & \text{if } l = t+1, \ldots, k \\ 2^{2^{n-l+1}} - 2^{2^{n-l}} & \text{if } l = k+1, \ldots, n \end{cases}$$
for some t. We omit the exact determination of the turnpoint t in order to save space. The conclusions drawn in Section 4 are independent of the exact value of this turnpoint.

The proof is omitted and can be found in [6].

An upper bound on the maximum number of nonterminal vertices in fore symmetry ordered OBDDs can be computed by adding the upper bounds on the W_l's just proven. The four cases of the lemma have to be distinguished. We only concentrate on one of these cases which we apply in the conclusions.

Case $k < 2^{2^{n-k-1}-1} - 2^{2^{n-k-2}-1}$

$$\begin{aligned} R_{fore}(n,k) &= \\ &= \sum_{l=1}^{k+1} W_l + \sum_{l=k+2}^{t} W_l + \sum_{l=t+1}^{n} W_l \\ &\leq \sum_{l=1}^{k+1} l + \sum_{l=k+1}^{t} (k+1)2^{l-k-1} + \sum_{l=t+1}^{n} \left(2^{2^{n-l+1}} - 2^{2^{n-l}}\right) \\ &= \frac{(k+1)(k+2)}{2} + (k+1)(2^{t-k} - 1) + 2^{2^{n-t}} - 2 \qquad (1) \end{aligned}$$

with $t = n - \lfloor \log_2(n-k+\log_2(k+1)) \rfloor + \epsilon$ for some $\epsilon \in \{0,1\}$.

Note that the least upper bound on OBDD sizes proved by Heap [4] is a special case of the upper bounds proved here.

3.2. Back symmetry ordered OBDDs

Assume w.l.o.g. that the variable order is fixed to $x_1, x_2, \ldots, x_n$ and $\lambda = \{x_{n-k+1}, \ldots, x_n\}$.

Lemma 2 The maximum number $R_{back}(n,k)$ of nonterminal vertices of an OBDD considered is given by $\sum_{l=1}^{n} W_l$ with

- For $1 \le l \le n-k$:
  $W_l \le min\{2^{l-1}, 2^{(k+1)2^{n-k-l+1}} - 2^{(k+1)2^{n-k-l}}\}$.
- For $n-k+1 \le l \le n$:
  $W_l \le min\{2^{n-k}(l+k-n), 2^{n-l+2}-2\}$.

Proof: As $W_{i+1} \le 2W_i$ for any $i > 1$ and $W_1 = 2^0$ hold, $W_l \le 2^{l-1}$. For $l \le n-k$, the number of Boolean functions which are defined on $\{x_l, \ldots, x_{n-k}, x_{n-k+1}, \ldots, x_n\}$ is $2^{(k+1)2^{n-k-l+1}}$ of which $2^{(k+1)2^{n-k-l}}$ do not depend on x_l and are not represented by x_l-vertices. This proves the first inequation.

For $l \ge n-k+1$, the number of nonconstant Boolean functions which are defined on $\{x_l, \ldots, x_n\}$ is $2^{n-l+2}-2$. On the other side, a Boolean function f with the property considered has at most $2^{n-k}(l-n+k)$ different cofactors with respect to $\{x_1, \ldots, x_{l-1}\}$. This proves the second inequation of the lemma. q.e.d.

The following lemma gives the term which determines W_l.

Theorem 2 For $2 \le n$ and $k \le n-1$ the following statements hold

1. If $n \le 3k+2$, W_l is determined by 2^{l-1} for $1 \le l \le n-k$.
2. If $n > 3k+2$, W_l is determined by
$$W_l \le \begin{cases} 2^{l-1} & , \text{ if } l = 1, \ldots, t \\ h(l) & , \text{ if } l = t+1, \ldots, n-k \end{cases}$$
with $h(l) := 2^{(k+1)2^{n-k-l+1}} - 2^{(k+1)2^{n-k-l}}$ and for some t. We omit the exact determination of the turnpoint t in order to save space. The conclusions drawn in Section 4 are independent of the exact value of this turnpoint.
3. If $2k \le n-1$, W_l is determined by $2^{n-l+2}-2$ for $n-k+1 \le l \le n$.
4. If $2k > n-1$, W_l is determined by
$$W_l \le \begin{cases} 2^{n-k}(l-n+k) & , \text{ if } l = n-k+1, \ldots, t \\ 2^{n-l+2}-2 & , \text{ if } l = t+1, \ldots, n \end{cases}$$
with $t = k - \lfloor \log_2(2k-n+1) \rfloor + \epsilon$ for some $\epsilon \in \{1,2\}$.

Proof:

1. $g_1(l) := 2^{l-1}$ is strictly increasing and $g_2(l) := 2^{(k+1)2^{n-k-l+1}} - 2^{(k+1)2^{n-k-l}}$ is strictly decreasing in l. It holds $g_1(n-k) \le g_2(n-k) \iff n \le 3k+2$.
2. Let $n > 3k+2$, $l \le n-k$, $g_1(l), g_2(l)$ defined as above, and $t = max\{l;\ 1 \le l \le n-k \text{ and } g_1(l) \le g_2(l)\}$. As $g_1(n-k) > g_2(n-k)$ and $g_1(1) < g_2(1)$, t exists and is unique.
3. $h_1(l) := 2^{n-k}(l-n+k)$ is increasing and $h_2(l) := 2^{n-l+2}-2$ is decreasing in l. It holds $h_2(n-k+1) \le h_1(n-k+1) \iff 2k \le n-1$.
4. Let $2k > n-1$, $l \ge n-k+1$, $h_1(l), h_2(l)$ defined as above, and $t = max\{l;\ n-k+1 \le l \le n \text{ and } h_1(l) \le h_2(l)\}$. As $h_2(n-k+1) > h_1(n-k+1)$ and $h_2(n) \le h_1(n)$ for $n \ge 2$, t exists and is unique.

   For $l = k - \lfloor \log_2(x) \rfloor + 1$ it can be proven that $h_2(l) \ge h_1(l)$ holds. $h_1(l) \ge h_2(l)$ holds for $l = k - \lfloor \log_2(x) \rfloor + 3$. This completes our proof.

q.e.d.

More details about this proof can be found in [6].

An upper bound on the maximum number of nonterminal vertices in back symmetry ordered OBDDs can be computed by adding the upper bounds on the W_l's just proven. Here, we only concentrate on the case which will be investigated further in the conclusions.

Case $2k > n-1$. Note, that it implies $n \le 3k+2$, too.

$$\begin{aligned} R_{back}(n,k) &= \sum_{l=1}^{n-k} W_l + \sum_{l=n-k+1}^{t} W_l + \sum_{l=t+1}^{n} W_l \\ &\le \sum_{l=1}^{n-k} 2^{l-1} + \sum_{l=n-k+1}^{t} 2^{n-k}(l-n+k) \\ &\quad + \sum_{l=t+1}^{n} (2^{n-l+2}-2) \\ &= 2^{n-k} - 1 + 2^{n-k-1}(t-n+k)(t-n+k+1) \\ &\quad -2(n-t) + 2^{n-t+2} - 4 \\ &= 2^{n-k} + 2^{n-k-1}(t-n+k)(t-n+k+1) \\ &\quad -2(n-t) + 2^{n-t+2} - 5 \qquad (2) \end{aligned}$$

for $t = k - \lfloor \log_2(2k-n+1) \rfloor + \epsilon$ for some $\epsilon \in \{1,2\}$.

4. Conclusions

We conjecture that the upper bounds proven are (not only) asymptotically tight. Working on this conjecture the

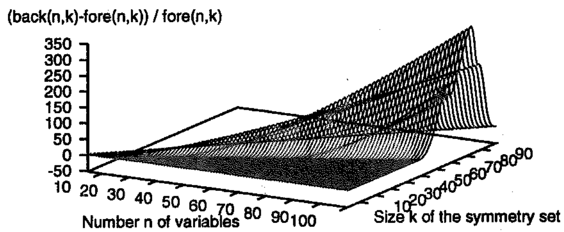

Figure 1. Relative difference of $R_{fore}(n,k)$ and $R_{back}(n,k)$

bounds set the trend where to locate the symmetric variables. Consider, e.g., $k = 3n/4$ with n large enough. The upper bound on the size of fore symmetry ordered OBDD is given by Equation 1 with $t = n - \lfloor \log_2(n/4 + \log_2(3n/4+1)) \rfloor + \epsilon$ for some $\epsilon \in \{0,1\}$

$$
\begin{aligned}
R_{fore}(n, 3n/4) &= \\
&= \Theta\left(\frac{n2^{n/4}}{n+\log_2(n)} + 2^{(n/4+\log_2(3n/4+1))2^{-s-\epsilon}}\right) \\
&= \Theta\left(\frac{n2^{n/4}}{n+\log_2(n)} + n^{2^{-s-\epsilon}} 2^{\frac{n}{4\cdot 2^{s+\epsilon}}}\right) \\
&= O\left(n2^{n/4}\right)
\end{aligned}
$$

with $s = \log_2(n/4 + \log_2(3n/4+1)) - \lfloor \log_2(n/4 + \log_2(3n/4+1)) \rfloor$, i.e., $0 \le s < 1$. The last equation holds because $s + \epsilon \ge 0$.

The upper bound on the size of back symmetry ordered OBDD is given by Equation 2 with $t = 3n/4 - \lfloor \log_2(n/2+1) \rfloor + \epsilon$ for some $\epsilon \in \{1,2\}$

$$
\begin{aligned}
R_{back}(n, 3n/4) &= \\
&= \Theta\left(2^{n/4} + n^2 2^{n/4} + n2^{n/4}\right) \\
&= \Theta\left(n^2 2^{n/4}\right).
\end{aligned}
$$

This set the trend to use fore symmetry ordered OBDDs for $k = 3n/4$.

To obtain an idea on the sizes of $R_{fore}(n,k)$ and $R_{back}(n,k)$ and where to locate the symmetry set of size k we have evaluated the formulas $R_{fore}(n,k)$ and $R_{back}(n,k)$ for $3 \le n \le 100$ by applying Lemma 1 and 2. Figure 1 shows the difference $R_{back}(n,k) - R_{fore}(n,k)$ with respect to $R_{fore}(n,k)$. For $k < n/2$, this value is small, sometimes positive, sometimes negative. For $k \ge n/2$, the difference is always positive which set the trend to use fore symmetry ordered OBDDs. The largest difference appears for k about $3n/4$.

Results concerning upper bounds on the size of symmetry ordered OBDDs where the block of symmetric variables are located at any position of the variable order are under work.

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