

MULTILEVEL GENERALIZATION OF RELAXATION ALGORITHMS FOR CIRCUIT SIMULATION

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We describe here a new strict multilevel generalization of relaxation techniques including Gauss-Seidel and Gauss-Jacobi algorithms used to solve algebraic circuit equations in Iterated Timing analysis, and differential systems by Waveform relaxation. The proposed modification of relaxation algorithms significantly enlarges the convergence region comparing with conventional (one-level) relaxation techniques without change the mode of circuit partitioning.

1. INTRODUCTION

The direct circuit simulation techniques used in programs like SPICE are known to become inefficient when applied to large systems. This arises from impossibility to pick timesteps independently for different state variables and overlinear CPU time spent growing along the increase of the circuit size and complexity [1]. As a promising way to overcome these problems the relaxation techniques were proposed that provide the almost linear dependence of CPU time on number of equations. Besides, dynamic variant of relaxation technique, waveform relaxation method (WR), allows the multirate integration for different subsystems of equations and appears to be particularly promising for parallel processing [1,2].

However, the implementation of relaxation techniques encounters often with some difficulties generated by slow convergence of iterations and/or by impossibility to partition the circuit without danger to lose the convergence entirely. These problems may occur when relaxation algorithms are used to solve systems of linear or nonlinear algebraic and differential equations. In addition, WR iterations often converges in a very nonuniform manner and

takes many iterations along all the simulation interval. In all these cases the convergence problems have the same nature that is the split matrix (or operator) corresponding to some relaxation iterations exceedingly differs by its spectral features from the Jacobi matrix (or operator for dynamic case) of the system to be solved.

Many attempts were made to improve the relaxation techniques [3,4] that dealt with optimal partitioning and ordering the subcircuits, time windowing, stepsize refinement, etc. but they didn't modify the relaxation algorithm itself. In this paper a new approach to convergence problems is presented that is based on multilevel generalization of relaxation iterations (MLR). It allows to distribute the mentioned spectral differences among all these iteration levels and to provide the convergence of iteration in each level. As a result, this approach significantly enlarges the convergence region of MLR respected to one-level relaxation method without change of the mode of circuit partitioning. Being the strict generalization of conventional relaxation algorithms, MLR inherit all positive features of the last such as linear time spent growing with the number of equations, the possibility of multirate integration for multilevel waveform relaxation algorithm (MLWR) and multiprocessor implementation. It also can be used together with other techniques applied to improve the characteristics of WR such as optimal partitioning and ordering the subcircuits, adaptive error control, time-step refinement, waveform Newton, etc.

2. BASIC RELAXATION ALGORITHM

We start the consideration with linear static problem that helps to understand the basic concept of multi-level technique. Let us suppose the algebraic system describing the circuit has the following form:

$F(X) \equiv AX + B = 0$, $A \in R^{N \times N}$; $X, B \in R^N$ (1)
and X_0 is the initial guess to exact solution X^* . To solve this problem the collection of iteration relaxation algorithms can be used that based on the structural decomposition of (1) such as Richardson method, Gauss-Seidel (GS), Gauss-Jacobi (GJ), sequential overrelaxation method and others that have the elements of split matrix $Q \in R^{N \times N}$ to be easily expressed from that of the Jacobian matrix A [5]. Let, for instance, A can be split in the form: $A = A_L + A_D + A_U$, where A_D - diagonal or block-diagonal matrix, A_L, A_U - the lower and upper (block) triangular matrices with zero diagonal elements. Then, in particular, $Q = \pm I_E$ (identity matrix) for Richardson method, $Q = A_D$ for GJ, $Q = A_L + A_D$ for GS, $Q = \tau^{-1} A_L + A_D$ for SOR method, where $\tau \in (0, 2)$ is relaxation parameter, etc. We'll use the term "Basic relaxation algorithm" (BR) for some technique that belongs to this set of algorithms. For given nonsingular Q the iterations of BR for system (1) can be represented in the form

$$-Q X^{i+1} = (-Q + A) X^i + B \quad (2)$$

where i denotes the iteration count. Note that the split matrix Q given the system (1) determines fully the convergence properties of iterations (2) and appropriating companion matrix as :

$$W = I_E - P, \text{ where } P = Q^{-1} A. \quad (3)$$

The iteration algorithm (BR) is known to converge if all the eigenvalues of W are contained in the unit ball of center (0, 0). Corresponding eigenvalues of P must locate inside the unit ball of center (1,0). Let the eigenvalues of P be inside the circle passing the points $(\alpha_{\min}, 0)$, $(\alpha_{\max}, 0)$, $\alpha_{\max} > \alpha_{\min} > 0$, of center $[(\alpha_{\min} + \alpha_{\max})/2, 0]$. Let OP denote the inner region of this circle. If all the eigenvalues $\sigma_j\{P\} \in OP$ then spectral radius of W , determining the convergence of BR, is:

$$\rho(W) \leq \max\{|1 - \alpha_{\min}|, |1 - \alpha_{\max}|\}. \quad (4)$$

If $\alpha_{\max} < 2$ then $\rho(W) < 1$ and BR converges.

3. MULTILEVEL RELAXATION ALGORITHM (MLR)

Considering (2) and (3), an important observation can be done that convergence problems of iterations (2) are produced by the difference between spectral characteristics of Q and A . Naturally, it is desirable to make some steps to reduce this inequality. We'll try to distribute this spectral difference among $L > 1$ iteration cycle loops.

Suppose that Q in (2) is replaced by $H_1 = q_1 Q + r_1 A$, where q_1, r_1 - some nonnegative coefficients. If H_1 approaches to Q then the iterations

$$-H_1 X^{i+1} = (-H_1 + A) X^i + B \quad (5)$$

acquire the same rate of convergence as (2). If H_1 approximates to A , then the rate of convergence in (5) increases and becomes infinite when $H_1 = A$. From other hand, the necessity appears in this case to solve (5) for X^{i+1} , so one iteration of (5) by its complexity becomes identical to primary problem (1).

The intermediate variant is of interest, when $q_1, r_1 > 0$, and the estimation for the spectral radius of corresponding companion matrix W_1 in (5) becomes less than in (4), i.e. the increase of the convergence rate in (5) is achieved by the use of inner iteration loop to solve the equations (5) for X^{i+1} . This partial problem can be considered as to solve the system with matrix H_1 . The process of the iteration cycles "granulation" can be continued to represent the solution of system (1) by $L > 1$ iteration cycle loops with companion matrices

$$W_m = I_E - H_m^{-1} H_{m-1} = I_E - (q_m I_E + r_m P)^{-1} \times \\ \times (q_{m-1} I_E + r_{m-1} P), \quad m = 1 \dots L. \quad (6)$$

The eigenvalues of W_m can be expressed via eigenvalues of P . Let $\alpha = \sigma_j\{P\}$, $\eta_m = \sigma_j\{W_m\}$, then

$$\eta_m = 1 - (q_{m-1} + r_{m-1} \alpha) / (q_m + r_m \alpha), \quad m = 1 \dots L. \quad (7)$$

If $q_m + r_m \alpha = \alpha^{(L-m)/L}$ for each $m=0\dots L$, (8)

then using (7) we get $\eta_1 = \eta_2 = \dots \eta_L = 1 - \alpha^{1/L}$. Defining q_m, r_m from the equations

$$q_m + r_m \alpha_{\{min, max\}} = \alpha_{\{min, max\}}^{(L-m)/L} \quad (9)$$

we make the upper (and lower) spectral bounds for W_m the same for all $m=1\dots L$. The conform mapping $P \rightarrow W_m$ transforms the region OP into the inner part of the circle traversing the points $(1 - \alpha_{min}^{1/L}, 0)$, $(1 - \alpha_{max}^{1/L}, 0)$ and having the center at the real axis. Consequently, the spectral radii of the companion matrices W_m , $m=1\dots L$ are defined as

$$\rho(W_m) \leq \max\{|1 - \alpha_{min}^{1/L}|, |1 - \alpha_{max}^{1/L}|\}, \quad (10)$$

that can be considered as generalization of (4). It follows from (10) that convergence domain of MLR respected to eigenvalues of P is characterized by the ball of center $(2^{L-1}, 0)$ and radius 2^{L-1} . This is also right for BR when $L=1$.

In each iteration level, as follows from (5), (6), we solve the system of the form:

$$H_{m-1}X + B_{m-1} = 0, \quad H_{m-1} \in R^{N \times N}; \quad X, B_{m-1} \in R^N \quad (11)$$

with split matrix H_m , where

$$H_m = q_m Q + r_m A, \quad B_m = -(H_m - H_{m-1})X + B_{m-1}. \quad (12)$$

From (9), (12) we obtain $H_0 = A$, $B_0 = B$, $H_L = Q$, and $0 = q_0 < q_1 < \dots < q_L = 1$, $1 = r_0 > r_1 > \dots > r_L = 0$.

Let $X_{(m)}^{i_m}$ denote the evaluation of X at i_m iteration that belongs to m -th iteration level. Let $t_m = q_m + r_m$, $H_m = t_m Q + r_m S$, where $S = A - Q$. Using (12) recursively, we get

$$\begin{aligned} B_{L-1} &= -(H_{L-1} - H_{L-2})X_{(L-1)}^{i_{L-1}} + B_{L-2} = \\ &= [(t_{L-2} - t_{L-1})Q + (r_{L-2} - r_{L-1})S]X_{(L-1)}^{i_{L-1}} + \dots \\ &\quad + [(t_0 - t_1)Q + (r_0 - r_1)S]X_{(1)}^{i_1} + t_0 B_0. \end{aligned} \quad (13)$$

Combining (11) and (13), we come to representation of L -level iteration process:

$$\begin{aligned} t_L Q X_{(L)}^{i_L+1} &+ [(t_{L-1} - t_L)Q + (r_{L-1} - r_L)S]X_{(L)}^{i_L} + \\ &+ [(t_{L-2} - t_{L-1})Q + (r_{L-2} - r_{L-1})S]X_{(L-1)}^{i_{L-1}} + \dots \\ &+ [(t_0 - t_1)Q + (r_0 - r_1)S]X_{(1)}^{i_1} + t_0 B_0 = 0. \end{aligned} \quad (14)$$

In particular, if $L=1$ then (14) gives

$$t_1 Q X^{i+1} + [(t_0 - t_1)Q + (r_0 - r_1)S]X^i + t_0 B_0 = 0, \quad (15)$$

that, accounting $t_0=1$, $r_0=1$, $r_1=0$, $t_1=1$, coincides with (2).

It must be added that easy and regular way was found to construct the equivalent circuit representation of MLR based on the corresponding equivalent circuit for BR. This technique is general and can be applied to linear and nonlinear static and dynamic circuits. For the sake of condensity this material is omitted though it is used as we consider the examples in Sec.6.

4. MLR FOR NONLINEAR CASE

The decomposition techniques (BR) are known to be used to solve the nonlinear system (1) if some continuation conditions are held. Because of MLR is the strict generalization of BR, it also can be used for this purpose. In order to represent MLR iterations in most general case, we need in additional denotations. Let $\bar{Q}, \bar{S} \in R^{N \times N}$ be pattern matrices for Jacobian parts of (1) Q and S respectively. All their elements belong to the set $\{0,1\}$, and $\bar{Q}_{ij}(\bar{S}_{ij}) = 1$ if and only if $Q_{ij}(S_{ij}) \neq 0$. In particular, the equality $\bar{Q}_{ij} + \bar{S}_{ij} = \{0,1\}$ takes place for GS and GJ iterations.

At first, we represent the nonlinear variant of (15) in the form of

$$F_v(Y_v) = 0, \quad v=1\dots N, \quad (16)$$

where F_v denotes the v -th equation of (1), $Y_v = t_1 \bar{Q}_v^t \bullet X^{i+1} + [(t_0 - t_1)\bar{Q}_v^t + (r_0 - r_1)\bar{S}_v^t] \bullet X^i$; $\bar{Q}_v^t, \bar{S}_v^t$ are the v -th transposed rows of $\bar{Q}, \bar{S}$ and symbol " $\bullet$ " means Schur matrix production [6]. It's easy to show that element-by-element representation of (16) leads to well-known expressions for GS, GJ, etc. Furthermore, linearizing of (16) gives (15).

The multilevel generalization of (16) can be written in similar way:

$$F_v(Y_v) = 0, \quad v=1\dots N, \quad (17)$$

where $Y_v = t_L \bar{Q}_v^t \bullet X_{(L)}^{i_L+1} + [(t_{L-1} - t_L) \bar{Q}_v^t + (r_{L-1} - r_L) \bar{S}_v^t] \bullet X_{(L)}^{i_L} + \dots + [(t_0 - t_1) \bar{Q}_v^t + (r_0 - r_1) \bar{S}_v^t] \bullet X_{(1)}^{i_1}$.

If $L=1$ then (17) gives (16) and linearizing of (17) leads to (14).

5. MULTILEVEL WAVEFORM RELAXATION METHOD

Let the node equations of the circuit can be represented in the form of differential system and/or its linear analog:

$$F(\dot{X}, X, t) \equiv C \dot{X} + AX + B = 0, \quad X_0(0) = X^*(0);$$

$$C, A \in R^{N \times N}; \quad X, B \in R^N, \quad (18)$$

with initial guess $X_0(t)$, where $F(\dot{X}, X, t)$ is

Lipschitz continuous with respect to $\dot{X}, X$ for all t . Suppose, for any "independent" node of the circuit there exists a chain of capacitances connecting this node to a reference one. In this case a Jacobian $[\partial F_k / \partial \omega_j]$ of function $F(\Omega, Z, t)$ as well as matrix C are diagonally dominant [7].

Let's introduce an operator $P(s) = (s\bar{Q} \bullet C + \bar{Q} \bullet A)^{-1} (sC + A)$ that is a dynamic analog of matrix P for MLR, $P_c = \lim_{s \rightarrow \infty} P(s) = (\bar{Q} \bullet C)^{-1} C$, $P_a = \lim_{s \rightarrow 0} P(s) = (\bar{Q} \bullet A)^{-1} A$. Let $\alpha_{c \min}, \alpha_{c \max} > 0$ be the bounds for spectra P_c such that all the eigenvalues of P_c lie inside the circle of center at the real axis traversing the points $(\alpha_{c \min}, 0), (\alpha_{c \max}, 0)$. Similarly, define the bounds $\alpha_{a \min}, \alpha_{a \max}$ for P_a .

As shown in [7, Theor. 5.3 and 5.4], the rate of WR iteration convergence in small-time region depends on the contraction degree in the

mapping $\dot{X}^{i+1} \leftarrow \dot{X}^i$. We can define the corresponding contraction factors via companion operator $\gamma_c = \lim_{s \rightarrow \infty} \rho(W(s)) = \rho(I_E - P_c) = \max\{|1 - \alpha_{c \min}|, |1 - \alpha_{c \max}|\}$. In the large-time region the contraction factor for the mapping $X^{i+1} \leftarrow X^i$ is: $\gamma_a = \lim_{s \rightarrow 0} \rho(W(s)) = \rho(I_E - P_a) = \max\{|1 - \alpha_{a \min}|, |1 - \alpha_{a \max}|\}$. The necessary condition of uniform convergence

for WR iterations expressed by inequality $\gamma_\infty < 1$, where $\gamma_\infty = \max_s \gamma(s) \geq \max(\gamma_c, \gamma_a)$ - the uniform contraction factor.

As a rule, in conventional ($L=1$) WR iterations, based on GS, GJ decomposition, only $\gamma_c < 1$ is guaranteed that follows from diagonal dominance in matrix C . In many cases MLWR makes it possible to ensure both the necessary conditions $\gamma_c, \gamma_a < 1$ for uniform convergence by reduction of $\gamma_{mc} = \lim_{s \rightarrow \infty} \rho(W_m(s))$, $\gamma_{ma} = \lim_{s \rightarrow 0} \rho(W_m(s))$, $m=1 \dots L$ as L increases.

We can represent the dynamic variant of MLR (MLWR) as:

$$F_v(\dot{Y}_v, Y_v, t) = 0, \quad v=1 \dots N, \quad (19)$$

$$\text{where } Y_v = \bar{Q}_v^t \bullet X_{(L)}^{i_L+1} + (r_{L-1} - r_L) \bar{S}_v^t \bullet X_{(L)}^{i_L} + \dots + (r_0 - r_1) \bar{S}_v^t \bullet X_{(1)}^{i_1}.$$

If $L=1$ then (19) gives well-known description for WR, and it produces also the multi-level iterations for nonlinear algebraic system when $\dot{X} \equiv 0, \dot{Y} \equiv 0$.

6. EXAMPLES AND EXPERIMENTAL RESULTS

Consider the examples of problems where conventional relaxation algorithms (BR) diverge as MLR ensures the convergence of iterations

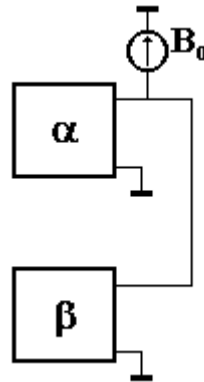


Fig. 1a

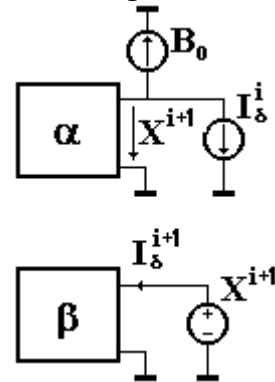


Fig. 1b

with the same type of circuit partitioning.

In the first example the static circuit (fig.1a) is decomposed by tearing the branches connecting

the subcircuits α and β . Fig.1b illustrates the equivalent circuit corresponding to BR. If subcircuits α , β have the admittance matrices Y_α , Y_β near the equilibrium condition then, using previous denotations, it's easy to show that $A = -Y_\alpha + Y_\beta$, $Q = Y_\alpha$, $S = Y_\beta$, $P = Q^{-1}A = I_E + Y_\alpha^{-1}Y_\beta$. Indeed, from fig.1b $Y_\alpha X^{i+1} + B_0 = -I_\beta^i$, $Y_\beta X^{i+1} = I_\beta^{i+1}$, so $X^{i+1} = -Y_\alpha^{-1}Y_\beta X^i - Y_\alpha^{-1}B_0$, the companion matrix $W = I_E - P = -Y_\alpha^{-1}Y_\beta$ and we come to above equalities. If subcircuits α, β are reciprocal then Y_α, Y_β are symmetric and positively defined. Hence, the eigenvalues of P locate inside the interval $(1, 1 + \rho^*)$, where $\rho^* = \rho(Y_\alpha^{-1}Y_\beta)$ and we can choose $\alpha_{min}=1$, $\alpha_{max}=1+\rho^*$. If $\rho^* > 1$ then $\rho(W) > 1$ and BR iterations diverge. In general, reordering the subcircuits is not a remedy because each eigenvalue $\sigma_j < 1$ of $Y_\alpha^{-1}Y_\beta$ produces an eigenvalue $\sigma_j > 1$ of $Y_\beta^{-1}Y_\alpha$.

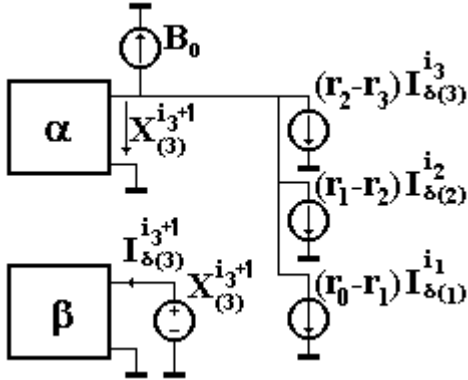


Fig.1c

To ensure the convergence, we construct MLR with $L > \log_2 \alpha_{max}$. If, for instance, $\alpha_{max}=5$, then minimum integer L is 3. From (9) we obtain $r_0=1$, $r_1=0.481$, $r_2=0.177$, $r_3=0$, $\rho(W_m)=0.71$, $m=1...3$. Fig 1c illustrates the circuit representation for MLR and it is a 3-level generalization of iterations in Fig.1b. The 14 iterations of outer level are sufficient to reduce the initial error by two orders.

In the second example the MOS circuit (fig.2a) is decomposed by block GS method. In each BR

iteration the first subcircuit (T_1, T_2) provides the voltage x_1^{i+1} , having x_2^i as an input. Then the second subcircuit (T_3, T_4, T_5) is analyzed for x_2^{i+1} , etc (see fig.2b). The detailed consideration gives $\alpha_{min}=1$, $\alpha_{max}=1+s_1s_5/(y_3(y_2+s_1))$, where

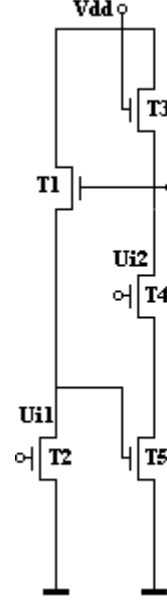


Fig.2a

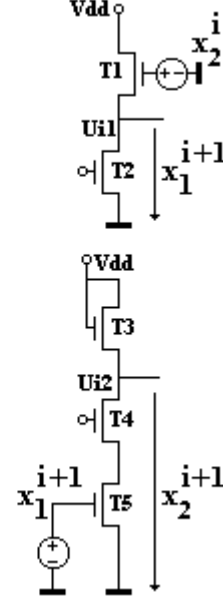


Fig.2b

s^* is a differential slope of transfer characteristic and y^* - an equivalent differential drain-source admittance for corresponding transistors. If the subcircuits are tightly coupled and $s_5 > y_3(1+y_2/s_1)$ then $\alpha_{max} > 2$ and BR iterations diverge. Furthermore, reordering the subcircuits

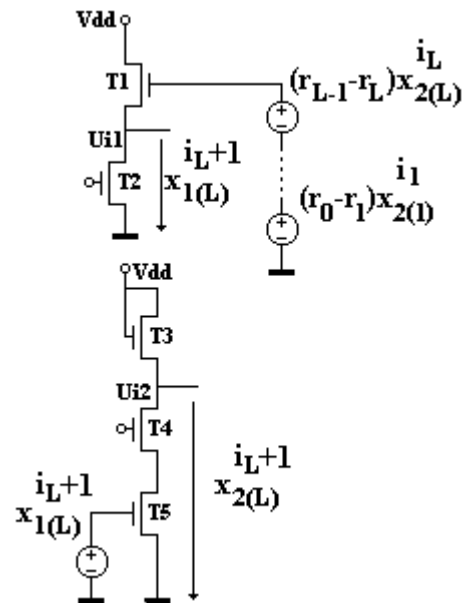


Fig.2c

does not alter $\alpha_{\min}, \alpha_{\max}$.

To make the iterations converging we can use MLR with $L > \log_2 \alpha_{\max}$ which ensures $\rho(W_m) = |1 - \alpha_{\max}^{1/L}| < 1$. The corresponding equivalent circuit is shown in fig.2c.

The next example is a transient analysis of TTL

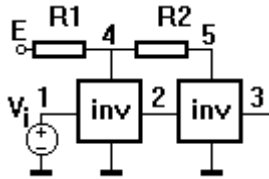


Fig.3a

bipolar inverter circuit with interconnect resistances (fig.3a) by WR and MLWR methods. Input is a constant voltage maintaining the inverters into active (amplifying) condition. This circuit is decomposed as shown in fig.3b, that corresponds to WR. Fig.3c demonstrates the first

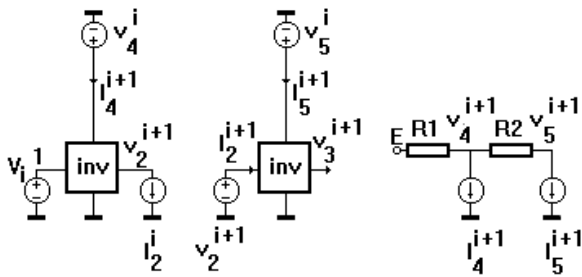
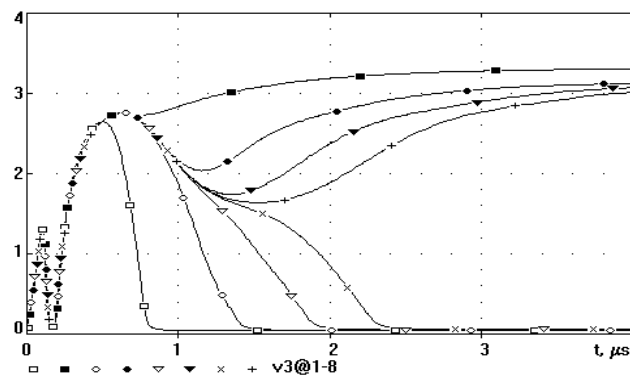


Fig.3b

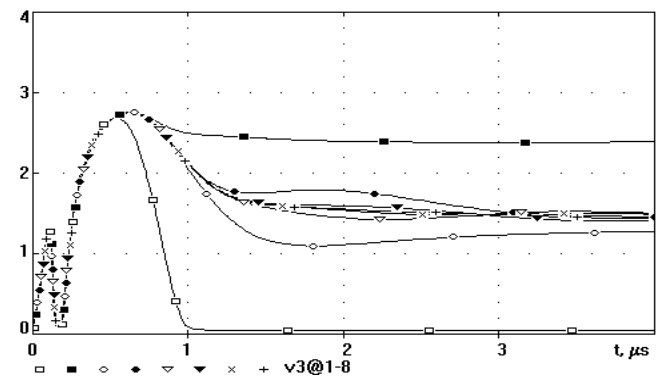
8 WR iterations (v3@1-8) that converge in a very nonuniform manner. Each one produces an



advance in a narrow interval (about $0.2\mu s$). This particularity implies the upper bound on the stepsize while analyzing the subcircuits and takes many iterations in a whole analysis

interval. At last, fig.3d illustrates the iterations of MLWR with $L=3, n_2=n_3=2$. Both contraction factors $\gamma_{mC}, \gamma_{mA}, m=1...L$ are less than 1, so the convergence is almost uniform. For given error, the number of MLWR iterations doesn't depend on the length of analysis interval.

As theory shows and was verified by



experiments, MLR and MLWR demonstrated the capability to improve the behavior of iterations when applied to linear algebraic system, to nonlinear discretized equations with Iterated Timing Algorithms and to differential equations with WR method.

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