

Analysis of Oscillator Injection Locking by Harmonic Balance Method

M.M. Gourary¹, S.G. Rusakov¹, S.L. Ulyanov¹, M.M. Zharov¹, B.J. Mulvaney², K.K. Gullapalli²

¹IPPM, Russian Academy of Sciences, 3 Sovetskaya str., Moscow, Russia, rusakov@ippm.ru

²Freescale Semiconductor Inc., 7700 W. Parmer Lane, Austin, Texas, USA, brian.mulvaney@freescale.com

Abstract

A new approach to analyze injection locking mode of oscillators under small external excitation is proposed. The proposed approach exploits existence conditions of the solution of HB linear system with degenerate matrix. The method allows one to obtain the locking range for an arbitrary oscillator circuit with an arbitrary periodic injection waveform. The approach can be easily implemented into a circuit simulator. Examples are given to confirm the correctness of the new approach.

1. Introduction

Injection locking is a phenomenon observed in oscillators perturbed by an external signal with frequency that is close to the frequency of free-running oscillations. The oscillators perturbed by these external signals are widely used in different applications [1-5].

Two main approaches are used to analyze injection locking of oscillators for small magnitude external excitation: the Adler equation and the Floquet-based phase equation.

The Adler equation [6] for an LC oscillator under small sinusoidal injection has the form

$$\frac{d\theta}{dt} = \omega_0 - \omega_{inj} - \frac{\omega_0}{2Q} \cdot \frac{V_{inj}}{V_{osc}} \sin(\theta) \quad (1)$$

where $\theta(t)$ is the slow-varying phase of the oscillator waveform with frequency ω_{inj} , ω_0 is the fundamental of the free-running oscillator, Q is the quality factor of the LC tank, and V_{inj} , V_{osc} are magnitudes of the injection signal and the capacitance voltage respectively. The injection locking condition can be easily obtained by steady-state solution of (1) with $d\theta/dt = 0$

$$|\omega_0 - \omega_{inj}| < \frac{\omega_0}{2Q} \cdot \frac{V_{inj}}{V_{osc}} \quad (2)$$

Equation (1) can be applied to LC oscillators only. A similar equation can be obtained for oscillators and injection-locked frequency dividers with a feedback loop including linear filter and amplifier [4, 5, 7, 8]. However this approach cannot be implemented into a circuit simulator to provide analysis of a general oscillator circuit.

The second approach is based on the phase equation from

Floquet theory [9]

$$\frac{d\alpha}{dt} = v^T(t + \alpha)b(t) \quad (3)$$

where $v(t)$ is a perturbation projection vector (PPV), $b(t)$ is the external excitation, and α is the unknown phase.

In comparison with (1) the important advantage of this phase equation is its applicability to an arbitrary oscillator circuit. Here locking conditions cannot be simply obtained by setting $d\alpha/dt = \text{const}$ because the solution of (3) corresponding with constant frequency does not exist. The straightforward application of (3) for capturing the injection locking effect requires multiple solutions of (3) by transient [10, 11] or harmonic balance [12] simulation. The analysis of injection locked frequency dividers by transient simulation of (3) is presented in [13].

A more efficient way to solve the problem is to determine the steady-state solution of the averaged equation obtained from (3). Such approach is considered in [14] to obtain the locking phase for sinusoidal excitation at the fundamental frequency. The explicit expression for locking range under the sinusoidal excitation is derived in [15]

$$\frac{|\Delta\omega|}{\omega_0} \leq \eta A \quad (4)$$

$$\eta = \max_{0 \leq \tau < 1} \left(\int_0^1 v\left(\frac{t+\tau}{f}\right) \cdot \sin(2\pi t) dt \right) \quad (5)$$

where f is the oscillation frequency, and A is an injection magnitude.

It is shown in [16] that (4, 5) being applied to an LC oscillator provides the same locking conditions as the Adler expression (2). An important advantage of (4, 5) is its applicability to an arbitrary oscillator circuit, which allows implementation of this approach into a circuit simulator. However the case of arbitrary (non-sinusoidal) periodic injection signal and the case of injection locked frequency dividers cannot be analyzed by (4, 5).

In this paper we present a new approach to analyze the injection locking mode of an oscillator. Unlike other methods the approach is not based on the phase differential equation. It is based on the existence condition of the solution of the linear system with degenerate matrix derived from harmonic balance (HB) formulation. The approach allows determination of the locking conditions of an

arbitrary oscillator excited by an arbitrary periodic waveform. The obtained conditions are presented in explicit form, and numerical solution of the differential equation is not required.

The paper is organized as follows. For the convenience of further treatment, section 2 presents basic equations of oscillator HB analysis. The principles of the proposed approach are explained in section 3 for the case of sinusoidal excitation at the fundamental frequency. The general case of arbitrary periodic excitation is considered in section 4. Comparison of locking range obtained by the proposed approach with simulation results is given in section 5.

2. Linearized oscillator HB equations

The oscillator is described by the following algebraic system in the frequency domain [17]

$$I(X) + j\omega K \cdot Q(X) = 0 \quad (6)$$

Here X , I , Q are the vectors of harmonics of nodal voltages, currents and charges respectively. Vectors X , I , Q contain components X_{kl} , I_{kl} , Q_{kl} , where k is a harmonic index, l is a nodal index. ω is the unknown fundamental frequency, K is a diagonal matrix of harmonic indexes

$$K = \text{diag}[\dots, -kE_N, \dots, -E_N, 0E_N, E_N, \dots, kE_N, \dots], \quad (7)$$

E_N is the unit matrix of the circuit size N .

In comparison with the case of nonautonomous circuits the system (6) has the additional variable because the fundamental frequency is unknown. Thus it is required to add the equation that provides an unambiguous solution. Usually fixing the phase of the first harmonic at the reference node is used that yields

$$\text{Im}(X_{1,r}) = 0 \quad (8)$$

The solution of (6, 8) can be obtained by the Newton method, which requires solving the linear augmented system at iterations

$$J \cdot \Delta X + jKQ \cdot \Delta\omega = R \quad (9)$$

$$\text{Im}(\Delta X_{1,r}) = 0 \quad (10)$$

The quasi-periodic model of a free-running oscillator under small periodic excitations is similar to the HB system for forced circuits [18] that describes linear periodically time-varying (LPTV) circuit

$$J(\Delta\omega) \cdot \Delta X = B \quad (11)$$

Here B is the vector of external small excitations, ΔX is the vector of small signal solution, $J(\Delta\omega)$ is a conversion matrix for the given frequency offset $\Delta\omega$

$$J(\Delta\omega) = G + j(\omega_0 \cdot K + \Delta\omega \cdot E)C = J_0 + j\Delta\omega \cdot C \quad (12)$$

where ω_0 is the oscillator fundamental, E is the unit matrix of the full system size, and G , C are block Toeplitz matrices of harmonics of nodal conductances and

capacitances.

The Jacobian matrix $J_0 = J(0) = G + j\omega_0 \cdot KC$ is singular and therefore there exists an eigenvector U corresponding to zero eigenvalue such that $J_0 \cdot U = 0$. It is known that U is the frequency domain representation of time derivative of the solution of (6, 8) [9]

$$U = \Gamma \frac{dx}{dt} = j\omega_0 K \cdot X \quad (13)$$

where Γ denotes the Fourier transform.

Also there exists an eigenvector V of the transposed Jacobian matrix corresponding to zero eigenvalue

$$J_0^T \cdot V = 0 \text{ or } V^T \cdot J_0 = 0 \quad (14)$$

The eigenvector V is the frequency domain representation of PPV from (3).

The vector V can be arbitrarily normalized. We shall use the following normalization condition

$$V^T C U = 1 \quad (15)$$

The condition (15) implies the condition

$$jV^T K Q = \frac{1}{\omega_0} \quad (16)$$

which results from the equality $j\omega_0 K \cdot Q = CU$, because both sides represent capacitance currents in the frequency domain

$$j\omega_0 K \cdot Q = \Gamma \frac{dq}{dt} = \Gamma \left(c(t) \frac{dx}{dt} \right) = C \cdot U \quad (17)$$

where $c(t) = \partial q / \partial x$ is the circuit capacitance matrix in the time domain.

3. Analysis of an oscillator under sinusoidal excitation at fundamental frequency

The solution of (11) demonstrates the unbounded growth when the offset frequency approaches to zero that contradicts the small signal assumption. Hence equation (11) is not valid for sufficiently small offset frequencies.

To investigate the behavior of an oscillator when the system (11) is not valid we consider the case of a small signal excitation at zero frequency offset. It is clear that there must exist a singletone solution with fundamental frequency ω_0 and harmonics $X + \Delta X$, where ΔX are deviations of harmonics of free running oscillator. The small signal system at fundamental frequency can be represented in the standard form (9) with $\Delta\omega = 0$

$$J_0 \cdot \Delta X = B \quad (18)$$

The augmenting equation (8) is not required because the frequency is known.

As mentioned above the matrix J_0 is singular. It is known that a linear system of equations with degenerate matrix either has no solution or has infinite set of solutions.

The last case occurs if the orthogonality condition holds, i.e.

$$V^T \cdot B = 0 \quad (19)$$

Now we take into account that there exist solutions of (6) and vectors V corresponding to different oscillator phase shifts ϕ . Hence we can define the phase shift such that (19) holds. Thus the existence condition of the solutions can be presented in the form of an equation with respect to the phase shift

$$V^T(\phi)B = 0 \quad (20)$$

where $V(\phi)$ is PPV corresponding to the solution with phase shift ϕ . Alternatively we may assume that the oscillator solution corresponds to zero phase shift, and the injection signal is shifted by $-\phi$. Then the condition (20) takes the following form

$$V^T \cdot B(-\phi) = 0 \quad (21)$$

Taking into account that the phase shift in the frequency domain is represented by the matrix multiplier $\exp(-jK\phi)$ we have

$$B(-\phi) = \exp(-jK\phi)B \quad (22)$$

where matrix K is defined in (7).

Then one can obtain from (21, 22)

$$V^T \exp(-jK\phi)B = 0 \quad (23)$$

Thus the solutions of (18) exist if the oscillator phase shift ϕ satisfies the expression (23).

In the case of a pure sinusoidal excitation of fundamental frequency applied to l -th node, the rhs vector of (18) is written as follows

$$B = \frac{1}{2}(b \cdot e_{1,l} + b^* \cdot e_{-1,l}) \quad (24)$$

Here e_{kl} is the unit vector selecting component (k, l) from rhs vector, b is the magnitude of the injection signal, and asterisk denotes complex conjugate value.

After substituting (24) into (23) one can obtain

$$V_{1l}b \exp(-j\phi) + V_{1l}^*b^* \exp(j\phi) = 0 \quad (25)$$

$$\text{or } \text{re}(V_{1l}b \exp(-j\phi)) = 0 \quad (26)$$

Let phasors V_{1l} , b be represented as

$$V_{1l} = |V_{1l}| \exp(j\varphi_{1l}) \quad b = |b| \exp(j\psi) \quad (27)$$

Then (26) is reduced to the equation with respect to ϕ

$$\cos(\varphi_{1l} + \psi - \phi) = 0 \quad (28)$$

Then the solution of (28) yields the locking phase

$$\phi_{lock} = \varphi_{1l} + \psi \pm \frac{\pi}{2} \quad (29)$$

After ϕ_{lock} is obtained the system to determine

harmonics deviations ΔX can be written in the form of (18) with corresponding phase shift

$$J_0 \cdot \Delta X = B(-\phi_{lock}) \quad (30)$$

where $B(-\phi_{lock})$ is defined by (22).

The infinite set of solutions of (30) evidences that linearized HB system is not sufficient to unambiguously determine the oscillator waveforms, and nonlinear effects must be taken into consideration. However many applications do not require the exact solution of the perturbed system. Often it is sufficient to approximate the solution by waveforms of free running oscillator with known phase shift that is provided by solving the phase equation (28) only.

4. Arbitrary periodic excitation at any offset frequency

When the frequency of the injected signal differs from the oscillator fundamental we can represent the linearized system in the form of (9) with known $\Delta\omega$

$$J_0 \Delta X + j\Delta\omega \cdot KQ = \frac{1}{2}B \quad (31)$$

where vector B possesses the complex conjugate property and zero DC value at each node

$$B_{-k,l} = B_{kl}^*, B_{0,l} = 0 \quad (32)$$

The phase shift providing the existence of solutions of (31) is obtained from the expression that can be derived in the same way as (23).

Equation (31) with shifted rhs is as follows

$$J_0 \Delta X + j\Delta\omega \cdot KQ = \frac{1}{2}B(-\phi) \quad (33)$$

Multiplying (33) by V^T and taking into account (22) and (14) we obtain

$$\frac{\Delta\omega}{\omega_0} - \frac{1}{2}V^T \exp(-jK\phi)B = 0 \quad (34)$$

In accordance with (32) and taking into account that vector V also possesses the complex conjugate property, the equation (34) can be represented in the real form

$$W(\phi) = \Delta\omega/\omega_0 \quad (35)$$

where

$$W(\phi) = \sum_{k>0} \sum_l |B_{kl}| |V_{kl}| \cos(\varphi_{kl} + \psi_{kl} - k\phi) \quad (36)$$

and phases and magnitudes are defined by the representation of phasors

$$V_{kl} = |V_{kl}| \exp(j\varphi_{kl}) \quad B_{kl} = |B_{kl}| \exp(j\psi_{kl}) \quad (37)$$

Equation (35) can be solved if the offset frequency belongs to the locking range

$$\min_{\phi} W(\phi) \leq \frac{\Delta\omega}{\omega_0} \leq \max_{\phi} W(\phi) \quad (38)$$

The general expression for locking range (38) corresponds with arbitrary periodic excitation when the injection waveform at l -th node is defined by Fourier coefficients B_{kl} . The locking range in this case can be easily obtained numerically by evaluating the maximum and minimum values $W(\phi)$, by sweeping phase shift ϕ in the range $0 - 2\pi$. If Fourier coefficients of injected signal are such that (38) holds then the locking phase can be evaluated by numerical solving of (35).

One can see that the proposed approach based on (35, 38) can be easily implemented in a circuit simulator. The eigenvector V (PPV) can be evaluated by the method proposed in [19] for HB representation of oscillator Jacobian matrix.

If only one sinusoidal waveform near k -th harmonic of the fundamental is injected into l -th node then (36) is reduced to

$$W(\phi) = |B_{kl}| |V_{kl}| \quad (39)$$

and the locking range (38) is defined as follows

$$\frac{|\Delta\omega|}{\omega_0} \leq |B_{kl}| |V_{kl}| \quad (40)$$

To compare (40) with the expression (4, 5) obtained by nonlinear phase macromodel [15] we perform the transformation of (5) taking into account that $\sin(2\pi t) = \text{Im}(e^{j2\pi t})$

$$\begin{aligned} \int_0^1 v\left(\frac{t+\tau}{f}\right) \sin(2\pi t) dt &= \text{Im} \int_0^1 v\left(\frac{t+\tau}{f}\right) e^{j2\pi t} dt = \\ &= \text{Im}(V_{1,l}^* e^{-j2\pi\tau}) = -|V_{1,l}| \sin(\phi_{1,l} + 2\pi\tau) \end{aligned} \quad (41)$$

Hence (5) can be evaluated as

$$\eta = \max_{0 \leq \tau < 1} (-|V_{1,l}| \sin(\phi_{1,l} + 2\pi\tau)) = |V_{1,l}|. \quad (42)$$

After substituting (42) into (4) one can see that locking range defined by (4) is equal to our evaluation (40) for the first harmonic ($k=1$, $A = |B_{1,l}|$). It is shown in [16] that for LC oscillator conditions (4, 5) provide the same result as the Adler condition (2). Thus for this simple case our result also coincides with the Adler condition.

The solution of (35) in the case of sinusoidal injection into one node is represented in the form

$$\phi_{lock} = \frac{\phi_{kl} + \psi_{kl} - \arccos\left(\frac{\Delta\omega}{\omega_0 |B_{kl}| |V_{kl}|}\right) - n\pi}{k}, \quad (43)$$

where $n = 0, \dots, k$.

Note, that the case $k > 1$ corresponds to a frequency divider with sinusoidal excitation.

5. Experimental results

We perform some numerical experiments to confirm that the injection locking conditions (36, 38) agree with transient simulation using a Spice-like simulator.

Experimental results are given for two oscillator circuit examples [20]: a Colpitts oscillator (fundamental frequency 3.39MHz) and the MC1648 oscillator (fundamental frequency 0.159MHz). The following values of first harmonics of PPV are obtained from steady-state solution: 4.03, 1.84 for Colpitts and MC1648 oscillators respectively.

The results of experiments are shown in Fig. 1, 2 for sinusoidal (a) and pulse train (b) injection signals. The locking region is shown in $\Delta\omega/\omega_0$ vs. A plane (Arnold tongue), where $\Delta\omega/\omega_0$ is relative frequency offset and A is the injection magnitude in mA. Solid lines present the boundary of locking region obtained by suggested approach (36, 38). The circles and squares correspond to Spice results with excitation by sinusoidal and pulse signals respectively. As we can see the obtained boundary is in good agreement with Spice results and practically coincides with them for small values of frequency offsets.

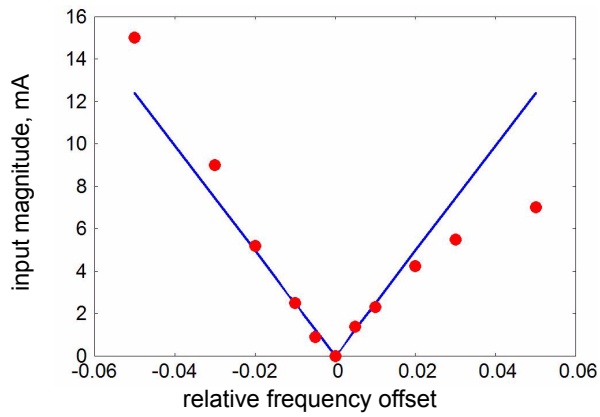
Similar results can be obtained for frequency dividers. The example of computed locking region for Colpitts oscillator in frequency divider mode (divide by 2) is shown in Fig. 3. Note that the expression (5) does not immediately provide the desired condition for frequency divider mode.

Fig. 4 illustrates the application of the suggested approach for a VCO based on two-stage BiCMOS ring oscillator (2.7 GHz) [1]. The each stage contains 6 bipolar and 5 MOS transistors. The application of the introduced locking conditions (36, 38) allows us to replace tedious Spice simulations, resulting in significant speedups in simulations. The values of frequency offset and amplitudes within the obtained boundary provide injection locking. For example two points were selected from the region (Fig. 4a): with -5% deviation from the center frequency and two different input amplitudes 27.5mV and 25.5mV. The first point is located on the boundary and the second one is below this boundary (Fig. 4a). The computed dependences of frequency on time are shown in Fig. 4b and Fig. 4c. Note the frequency locking in the first case and the absence of frequency locking in the second case.

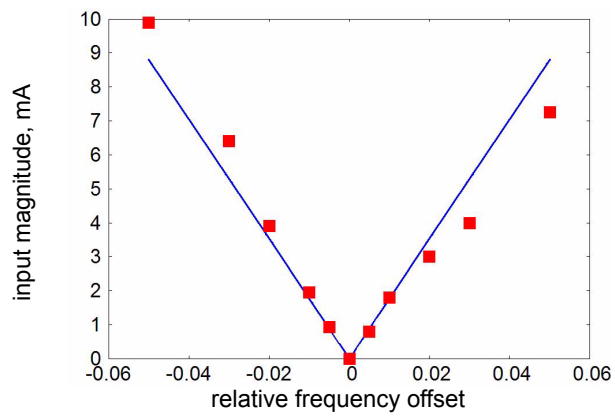
6. Conclusion

A new approach to analyze injection locking conditions of arbitrary oscillator circuits is proposed. The approach is based on the rigorous existence conditions of solution of HB linear system with degenerate matrix.

Unlike other methods the proposed approach determines locking conditions for an arbitrary periodic injection waveform. Numerical experiments confirm the correctness of the proposed approach.

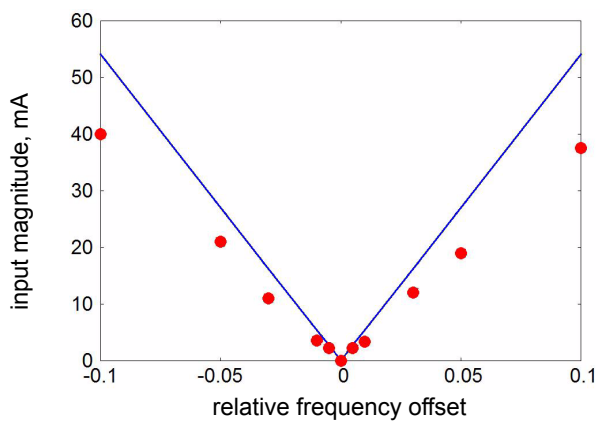


a)

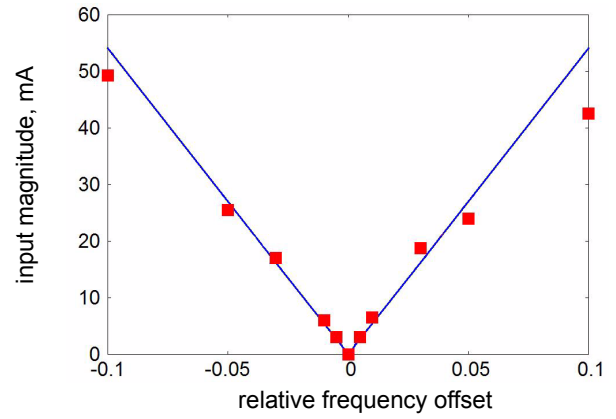


b)

Figure 1. Locking region for Colpitts oscillator perturbed by sinusoidal (a) and pulse (b) signal



a)



b)

Figure 2. Locking region for MC1648 oscillator perturbed by sinusoidal (a) and pulse (b) signal

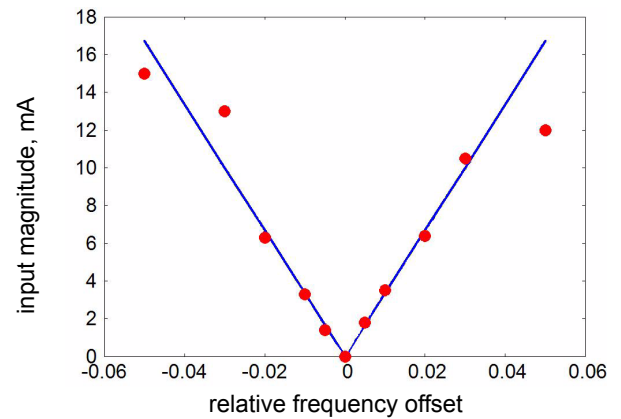
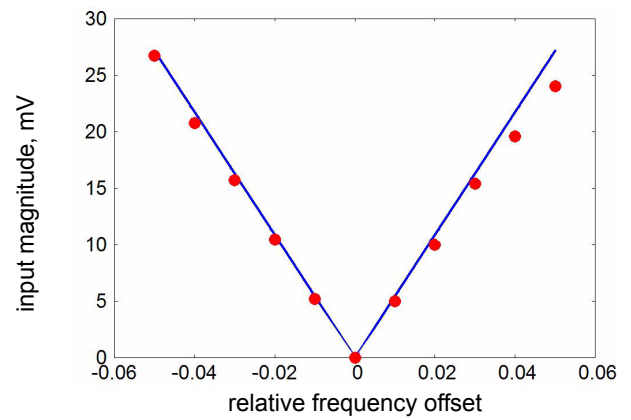


Figure 3. Locking region for Colpitts oscillator in the frequency divider mode perturbed by sinusoidal signal.



a)

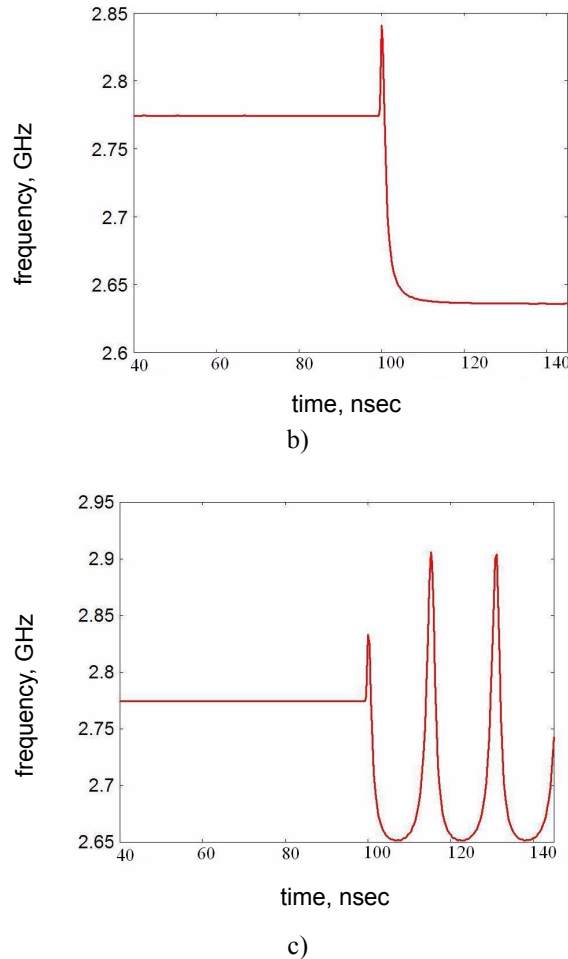


Figure 4. Locking region (a), frequency vs time (b,c) for BiCMOS oscillator

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